Riemann's Hypothesis as an Eigenvalue Problem. III

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ABSTRACT

We give conditional induction proofs for the existence of a small zero-free strip inside the critical strip of Riemann's zeta function $\zeta(s)$. The starting point is some formulas for the eigenvalues λ of certain matrices A_N over the integers, whose determinants are connected with Riemann's hypothesis by the equation $\det A_N = N!\Sigma_{1 \leqslant n \leqslant N}\mu(n)/n$, where μ denotes the Möbius function. The conditions of the proofs refer to properties of the characteristic polynomials $\chi_N(x)$ of the matrices A_N near $x = 0$ and/or the existence of small eigenvalues. A typical example: If for every $N \geqslant N_0$ at least one of the polynomials $\chi_M(x)$, $N \leqslant M \leqslant N + N^{1-\varepsilon}$, has a zero λ such that $-0.09 \leqslant \lambda \leqslant 1.04$, then $\zeta(s) \neq 0$ if $\operatorname{Re} s > 1 - \varepsilon$.

0. INTRODUCTION

The purpose of this paper is to prove the following conditional estimates:

$$\hat{\chi}_N(0) = \sum_{1 \leqslant n \leqslant N} \frac{\mu(n)}{n} = \mathrm{O}(N^{-\varepsilon}), \tag{0.1}$$

and, what is equivalent (via partial summation; cf. [6, Theorem 4]),

$$\hat{\chi}_N(1) = \sum_{1 \leqslant n \leqslant N} \frac{\rho(n)}{n} = \mathrm{O}(N^{-\varepsilon}). \tag{0.2}$$

μ denotes the Möbius function, and ρ is multiplicative with

$$\rho(p^m) = (-p)^m \prod_{1 \leqslant k \leqslant m} (p^k - 1)^{-1} \tag{0.3}$$

for prime powers p^m.

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Estimates such as (0.1) and (0.2) imply the existence of a small zero-free strip inside the critical strip of Riemann's zeta function $\zeta(s)$. In particular,

$$\hat{\chi}_N(0) = O(N^{-1/2+\varepsilon}) \qquad \text{for every} \quad \varepsilon > 0$$

is equivalent to the Riemann hypothesis [8, Theorem 14.25(C); 5, Theorem 1].

The unproved conditions under which (0.1) and (0.2) are shown here are assumptions concerning the characteristic polynomials $\chi_N(x)$, and in normalized form

$$\hat{\chi}_N(x) = \chi_N(x) \prod_{2 \leqslant m \leqslant N} (x-m)^{-1},$$

of the matrices

$$A_N = (a_{m,n})_{2 \leqslant m,n \leqslant N}, \qquad a_{m,n} = \begin{cases} m-1 & \text{if} \quad m|n, \\ -1 & \text{if} \quad m \nmid n, \end{cases}$$

and assumptions on the position of the eigenvalues $\lambda_1, \ldots, \lambda_{N-1}$ of the matrices A_N. More precisely, these conditions refer to properties of $\chi_N(x)$ near $x = 0$ and/or the existence of small eigenvalues λ of A_N.

Section 1 repeats results of [5] and [6] and establishes equations for the inductive proofs of the following sections.

In Section 2 we assume the existence of small eigenvalues to prove (0.1) and (0.2). A typical result is, for sufficiently small $\varepsilon > 0$ (Theorem 7): If for every $N \geqslant N_0$ at least one of the polynomials $\chi_M(x)$, $N \leqslant M \leqslant N + N^{1-\varepsilon}$, has a zero λ such that $-0.09 \leqslant \lambda \leqslant 1.04$, then $\hat{\chi}_N(0) = O(N^{-\varepsilon})$ and $\hat{\chi}_N(1) = O(N^{-\varepsilon})$.

In Section 3 we discuss the sequence of quotients

$$\gamma_N := \frac{\hat{\chi}_N(0)}{\hat{\chi}_N(1)}, \qquad N = 2, 3, \ldots. \tag{0.4}$$

This sequence probably does not converge, but Theorem 8 suggests that its elements should on average lie near

$$\zeta_\infty := \prod_{m \geqslant 2} \zeta(m) = 2.294856\ldots.$$

Then we assume irregularities of this sequence and prove (0.1) and (0.2). A typical result (Theorem 9(1)): If $-13.706 \leqslant \gamma_N \leqslant 0.482$ sufficiently often, then $\hat{\chi}_N(0) = O(N^{-\varepsilon})$.

In Section 4 we assume irregularities in the sequences

$$\kappa_N := \frac{\hat{\chi}_N'(0)}{\hat{\chi}_N(0)}, \quad \sigma_N := \frac{\hat{\chi}_N'(1)}{\hat{\chi}_N(1)}, \qquad N = 2,3,\ldots, \tag{0.5}$$

and then do the same (Corollary of Theorem 11):

If $|\kappa_N| \geqslant 0.520$ sufficiently often, then $\hat{\chi}_N(0) = O(N^{-\varepsilon})$.
If $|\sigma_N| \geqslant 1.990$ sufficiently often, then $\hat{\chi}_N(1) = O(N^{-\varepsilon})$.

Here again it seems unlikely that $(\kappa_N)_{N>1}$ or $(\sigma_N)_{N>1}$ converges, but κ_N should on average lie near

$$\frac{1}{\zeta(2)} - 1 = -0.392073\ldots,$$

and σ_N near

$$\sum_{n \geqslant 2} \left(1 - \prod_{m \geqslant n} \zeta(m)\right) = -1.989548\ldots$$

(Theorem 11). The definition (0.5) of κ_N and σ_N refers to $\hat{\chi}_N(x)$ and its derivative at $x = 0$ and $x = 1$, so that irregularities of $(\kappa_N)_{N>1}$ and $(\sigma_N)_{N>1}$ can be explained by the existence or nonexistence of small eigenvalues of the matrices A_N, for we have

$$\kappa_N = \sum_{2 \leqslant n \leqslant N} \frac{1}{n} - \sum_{\lambda} \frac{1}{\lambda},$$

$$\sigma_N = \sum_{2 \leqslant n \leqslant N} \frac{1}{n-1} - \sum_{\lambda} \frac{1}{\lambda - 1},$$

where λ runs through all eigenvalues of A_N.

It may be noticed that the investigations are concentrated at the points $x = 0$ and $x = 1$ of $\chi_N(x)$. The reason is: A recursive expansion of the determinant of the matrix $xI - A_N$ yields $\hat{\chi}_N(x) - \hat{\chi}_{N-1}(x) = \hat{h}_N(x)$ for the normalized characteristic polynomial [5, Theorem 3(3)] with $\hat{h}_n(0) = \mu(n)/n$,

$\hat{h}_n(1) = \rho(n)/n$ [thus proving the left equations in (0.1) and (0.2)], and the functions $\hat{h}_N(x)$ are multiplicative in N if and only if $x = 0$ or $x = 1$ [6, Theorem 1(1)]. Moreover the point $x = 1$ seems to be more appropriate for investigations than $x = 0$, for $x = 1$ reflects in a simple way all the polynomials $\chi_N(x)$:

$$\binom{N}{n}\chi_N(n) = (-1)^{N-n}N!\frac{\tau(n)}{n^2}\hat{\chi}_{[N/n]}(1), \qquad 1 \leqslant n \leqslant N, \tag{0.6}$$

[5, Theorem 4] and

$$\hat{\chi}_N(x) = (1-x)\sum_{1\leqslant n\leqslant N}\frac{\tau(n)}{n(n-x)}\hat{\chi}_{[N/n]}(1) \tag{0.7}$$

[6, Theorem 5(2)]. The function τ is multiplicative,

$$\tau(p^m) = \prod_{1\leqslant k\leqslant m}(1-p^{-k})^{-1}, \tag{0.8}$$

and [6, Theorem 3(1)]

$$\sum_{n\geqslant 1}\frac{\tau(n)}{n^s} = \prod_{m\geqslant 0}\zeta(s+m). \tag{0.9}$$

ρ is related to τ as μ is to the constant function 1:

$$\sum_{d\mid n}\rho(d)\tau\left(\frac{n}{d}\right) = \delta_{1,n} \tag{0.10}$$

[6, Theorem 2], and hence

$$\left(\sum_{n\geqslant 1}\frac{\rho(n)}{n^s}\right)\left(\sum_{n\geqslant 1}\frac{\tau(n)}{n^s}\right) = 1. \tag{0.11}$$

The conditions of the theorems in the Sections 2–4 can be regarded as generalizations of conditions on sign changes (for instance, $\gamma_N \leqslant 0.177$ in Theorem 9(3) of Section 3: $\gamma_N < 0$ would imply that $\hat{\chi}_N(0)$ and $\hat{\chi}_N(1)$ have different signs, i.e. that there is an eigenvalue of A_N in the interval $[0, 1]$, which is a condition of the type in Section 2). The study of sign changes in connection with the Riemann hypothesis has a long tradition, starting with Gauss's conjecture and Riemann's statement in his famous paper [4] of 1859 that for $x > 2$, one has $\Delta_1(x) := \pi(x) - \operatorname{li} x < 0$ (cf. [2]). More intimately connected [for instance via (0.4)] with our conditions are the sign changes of

$$M(x) = \sum_{n \leqslant x} \mu(n) \quad \text{and} \quad M_0(x) = \sum_{n \leqslant x} \frac{\mu(n)}{n},$$

which have been studied by Kátai [1] and Pintz [3]. These papers indicate that the conditions here must be truly deep.

This may suggest that examining $\chi_N(x)$ for small x is closely related to the classical study of the Riemann zeta function. Therefore it should be mentioned that it is by no means only the study of $\chi_N(x)$ near 0 that can exhibit appropriate information for proofs of the existence of zero-free strips for $\zeta(s)$. It certainly is unlikely that any knowledge about the large eigenvalues (as for instance in Theorem 6 of [6]) can furnish enough information to prove estimates like (0.1) or (0.2). But appropriate information on the real eigenvalues of A_N in the neighborhood of $\sqrt{N}$ can lead to proofs of (0.1) and (0.2): The "normal" situation is [5, Theorem 5] that each interval $[n, n+1[$, $1 \leqslant n \leqslant N-1$, contains exactly one eigenvalue of A_N. But if, for sufficiently many numbers N, there are intervals $[n, n+1[$ near $\sqrt{N}$ without eigenvalues of A_N, then (0.2) is true. And the nonexistence of eigenvalues in $[n, n+1[$ results from the existence of pairs of complex eigenvalues, which in turn can be caused by sign changes of the sequence $(\hat{\chi}_N(1))_{N>1}$ (cf. the introduction of [6]).

The Appendix, which is partly due to H.-J. Toussaint, contains four tables with numerical examples illustrating the theorems of Sections 2–4.

Numerical constants:

$$\zeta_\infty := \sum_{n \geqslant 1} \frac{\tau(n)}{n^2} = \prod_{m \geqslant 2} \zeta(m) = 2.294856\ldots,$$

$$\rho_\infty := \sum_{n \geqslant 1} \frac{|\rho(n)|}{n^2} = \prod_{m \geqslant 2} \frac{\zeta(m)}{\zeta(2m)} = 2.072956\ldots,$$

$$\tau_\infty := \sum_{n \geqslant 2} \frac{\tau(n)}{(n-1)n} = \sum_{k \geqslant 1} \Big(\prod_{m > k} \zeta(m) - 1 \Big) = 1.989548\ldots.$$

1. INDUCTION FORMULAE

The equations

$$\hat{\chi}_N(1) = \sum_{1 \leqslant n \leqslant N} \frac{\rho(n)}{n^2} \hat{\chi}_{[N/n]}(0), \tag{1.1}$$

$$\hat{\chi}_N(0) = \sum_{1 \leqslant n \leqslant N} \frac{\tau(n)}{n^2} \hat{\chi}_{[N/n]}(1), \tag{1.2}$$

[6, Theorem 4], which connect the values of the characteristic polynomials $\chi_N(x)$ at $x = 0$ and $x = 1$, can be interpreted as limits of similar equations which arise when we pass step by step from

$$\sum_{d \mid n} \mu(d) = \delta_{1,n} \tag{1.3}$$

to

$$\sum_{d \mid n} \rho(d) \tau\left(\frac{n}{d}\right) = \delta_{1,n} \tag{1.4}$$

[6, Theorem 2], via the functions

$$\mu^{(1)}(n) := \mu(n), \qquad \mu^{(k+1)}(n) := \sum_{a,b \geqslant 1,\, ab = n} \mu^{(k)}(a) \mu^{(1)}(b) b^{-k},$$

$$\tau^{(1)}(n) := 1, \qquad \tau^{(k+1)}(n) := \sum_{a,b \geqslant 1,\, ab = n} \tau^{(k)}(a) \tau^{(1)}(b) b^{-k}.$$

Induction on k, starting with (1.3), shows immediately

$$\sum_{d \mid n} \mu^{(k)}(d) \tau^{(k)}\left(\frac{n}{d}\right) = \delta_{1,n}, \tag{1.5}$$

and we have

$$\lim_{k \to \infty} \mu^{(k)} = \rho, \qquad \lim_{k \to \infty} \tau^{(k)} = \tau; \tag{1.6}$$

hence (1.4) is the limit of (1.5) for $k \to \infty$. To prove (1.6) we notice that by definition of $\mu^{(k)}$,

$$\sum_{n \geqslant 1} \frac{\mu^{(k+1)}(n)}{n^s} = \left(\sum_{n \geqslant 1} \frac{\mu^{(k)}(n)}{n^s} \right) \left(\sum_{n \geqslant 1} \frac{\mu(n)}{n^{s+k}} \right),$$

which yields by induction

$$\sum_{n \geqslant 1} \frac{\mu^{(k+1)}(n)}{n^s} = \prod_{0 \leqslant m \leqslant k} \zeta(s+m)^{-1} \tag{1.7}$$

and, by (0.9) and (0.11),

$$\sum_{n \geqslant 1} \left(\lim_{k \to \infty} \mu^{(k)}(n) \right) \frac{1}{n^s} = \prod_{m \geqslant 0} \zeta(s+m)^{-1} = \sum_{n \geqslant 1} \frac{\rho(n)}{n^s}.$$

This implies equality for the coefficients of the Dirichlet series, which is the first assertion in (1.6). The second one follows in exactly the same way.

THEOREM 1. *For all $k \geqslant 1$,*

(1) $$\sum_{1 \leqslant n \leqslant N} \frac{\tau(n)}{n^{k+2}} \hat{\chi}_{[N/n]}(1) = \sum_{1 \leqslant n \leqslant N} \frac{u^{(k)}(n)}{n^2} \hat{\chi}_{[N/n]}(0),$$

(2) $$\sum_{1 \leqslant n \leqslant N} \frac{\rho(n)}{n^{k+2}} \hat{\chi}_{[N/n]}(0) = \sum_{1 \leqslant n \leqslant N} \frac{\tau^{(k)}(n)}{n^2} \hat{\chi}_{[N/n]}(1),$$

and (1.1), (1.2) *result from these equations for $k \to \infty$.*

Proof. Equation (1.1) shows

$$\sum_{1 \leqslant n \leqslant N} \frac{\tau(n)}{n^{k+2}} \hat{\chi}_{[N/n]}(1) = \sum_{1 \leqslant n \leqslant N} \frac{\tau(n)}{n^{k+2}} \left(\sum_{1 \leqslant m \leqslant N/n} \frac{\rho(m)}{m^2} \hat{\chi}_{[N/mn]}(0) \right)$$

$$= \sum_{1 \leqslant r \leqslant N} \frac{1}{r^2} \hat{\chi}_{[N/r]}(0) \left(\sum_{mn = r} \frac{\tau(n)}{n^k} \rho(m) \right),$$

and

$$\sum_{mn=r} \frac{\tau(n)}{n^k}\rho(m) = \mu^{(k)}(r),$$

since

$$\begin{aligned}\sum_{r\geqslant 1} \frac{1}{r^s}\left(\sum_{mn=r} \frac{\tau(n)}{n^k}\rho(m)\right) &= \left(\sum_{n\geqslant 1} \frac{\tau(n)}{n^{s+k}}\right)\left(\sum_{n\geqslant 1} \frac{\rho(n)}{n^s}\right)\\ &= \prod_{0\leqslant m\leqslant k-1} \zeta(s+m)^{-1} \qquad \text{[by (0.9) and (0.11)]}\\ &= \sum_{r\geqslant 1} \frac{\mu^{(k)}(r)}{r^s} \qquad \text{[by (1.7)]}.\end{aligned}$$

The second equation follows in exactly the same way. ■

THEOREM 2 (Interpolation theorem). *Suppose that $\lambda_1,\ldots,\lambda_M$ are zeros of $\chi_N(x)$, which need not be different. Then for $\delta = 0$ and $\delta = 1$,*

$$\begin{aligned}\hat{\chi}_N(\delta) = \frac{1}{(M-\delta)!}&\left(\prod_{j=1}^{M}(\lambda_j-\delta)\right)\\ &\times \sum_{M+1\leqslant n\leqslant N}\left(\prod_{j=1}^{M}\frac{n-j}{n-\lambda_j}\right)\frac{\tau(n)}{(n-\delta)n}\hat{\chi}_{[N/n]}(1).\end{aligned}$$

Proof. Lagrange interpolation of the polynomial $\chi_N(x)\prod_{j=1}^{M}(x-\lambda_j)^{-1}$ of degree $N-M-1$ at $n = M+1, M+2,\ldots,N$ shows

$$\chi_N(x)\prod_{j=1}^{M}(x-\lambda_j)^{-1} = \sum_{M+1\leqslant n\leqslant N}\frac{\chi_N(n)}{\prod_{1\leqslant j\leqslant M}(n-\lambda_j)}\prod_{\substack{M+1\leqslant m\leqslant N\\ m\neq n}}\frac{m-x}{m-n},$$

and by (0.6),

$$\chi_N(n) = (-1)^{N-n}(N-n)!n!\frac{\tau(n)}{n^2}\hat{\chi}_{[N/n]}(1).$$

Hence

$$\chi_N(x) = (-1)^{N-1}\left(\prod_{j=1}^{M}(\lambda_j - x)\right)$$
$$\times \sum_{M+1 \leqslant n \leqslant N}\left(\prod_{j=1}^{M}\frac{n-j}{n-\lambda_j}\right)\left(\prod_{\substack{m=M+1\\ m\neq n}}^{N}(m-x)\right)\frac{\tau(n)}{n}\hat{\chi}_{[N/n]}(1),$$

and, by definition of $\hat{\chi}_N(x)$,

$$\chi_N(0) = (-1)^{N-1}N!\hat{\chi}_N(0), \qquad \chi_N(1) = (-1)^{N-1}(N-1)!\hat{\chi}_N(1).$$

■

THEOREM 3. *For all x in a δ-neighborhood of 0, $\delta < 1$, where $\chi_N(x)$ has no zeros,*

$$\hat{\chi}_N(x) = \hat{\chi}_N(0)\exp\left(\sum_{k\geqslant 1}\frac{1}{k}\kappa_{N,k}x^k\right)$$

with

$$\kappa_{N,k} = \sum_{2\leqslant n\leqslant N} n^{-k} - \sum_{\lambda}\lambda^{-k},$$

and λ runs through all zeros of $\chi_N(x)$.

Proof.

$$\frac{d}{dx}\left(\prod_{2\leqslant m\leqslant N}(x-m)\right) = \left(\prod_{2\leqslant m\leqslant N}(x-m)\right)\sum_{2\leqslant m\leqslant N}\frac{1}{x-m},$$

$$\frac{d}{dx}\left(\prod_{\lambda}(x-\lambda)\right) = \left(\prod_{\lambda}(x-\lambda)\right)\sum_{\lambda}\frac{1}{x-\lambda},$$

and hence, by definition of $\hat{\chi}_N(x)$,

$$\frac{d}{dx}\log|\hat{\chi}_N(x)| = \frac{\hat{\chi}_N'(x)}{\hat{\chi}_N(x)}$$

$$= \frac{\chi_N'(x)}{\chi_N(x)} - \frac{\frac{d}{dx}\left(\prod_{2\leqslant m\leqslant N}(x-m)\right)}{\prod_{2\leqslant m\leqslant N}(x-m)}$$

$$= \sum_{2\leqslant m\leqslant N}\frac{1}{m-x} - \sum_{\lambda}\frac{1}{\lambda-x}$$

$$= \sum_{2\leqslant m\leqslant N}\frac{1}{m}\left\{\sum_{k\geqslant 0}\left(\frac{x}{m}\right)^k\right\} - \sum_{\lambda}\frac{1}{\lambda}\left\{\sum_{k\geqslant 0}\left(\frac{x}{\lambda}\right)^k\right\}$$

$$= \sum_{k\geqslant 1}\kappa_{N,k}x^{k-1}.$$

Now integration yields

$$\log|\hat{\chi}_N(x)| = \sum_{k\geqslant 1}\frac{1}{k}\kappa_{N,k}x^k + C;$$

hence

$$|\hat{\chi}_N(x)| = e^C\exp\left(\sum_{k\geqslant 1}\frac{1}{k}\kappa_{N,k}x^k\right),$$

and $x = 0$ shows $|\hat{\chi}_N(0)| = e^C$. ∎

THEOREM 4. *Suppose that $\hat{\chi}_N(0) \neq 0$. Then for all $k \geqslant 1$,*

$$\sum_{1\leqslant n\leqslant N}\frac{\tau(n)}{n^{k+2}}\hat{\chi}_{[N/n]}(1) = F_{N,k}\hat{\chi}_N(0).$$

and the coefficients are determined by the equation

$$\sum_{k \geqslant 0} F_{N,k} x^k = \prod_{k \geqslant 1} \left\{ \sum_{\nu \geqslant 0} \frac{1}{\nu!} \left(\frac{1}{k} (\kappa_{N,k} + 1) \right)^{\nu} x^{k\nu} \right\}.$$

In particular, with $K_{N,k} = \kappa_{N,k} + 1$,

$$\begin{aligned} F_{N,0} &= 1, \\ F_{N,1} &= K_{N,1}, \\ F_{N,2} &= \tfrac{1}{2}\left(K_{N,1}^2 + K_{N,2}\right), \\ F_{N,3} &= \tfrac{1}{6}\left(K_{N,1}^3 + 3K_{N,1}K_{N,2} + 2K_{N,3}\right). \end{aligned}$$

Proof. On one hand we have by (0.7)

$$\begin{aligned} \frac{\hat{\chi}_N(x)}{1-x} &= \sum_{1 \leqslant n \leqslant N} \frac{\tau(n)}{n(n-x)} \hat{\chi}_{[N/n]}(1) \\ &= \sum_{1 \leqslant n \leqslant N} \frac{\tau(n)}{n^2} \left\{ \sum_{k \geqslant 0} \left(\frac{x}{n} \right)^k \right\} \hat{\chi}_{[N/n]}(1) \\ &= \sum_{k \geqslant 0} \left(\sum_{1 \leqslant n \leqslant N} \frac{\tau(n)}{n^{k+2}} \hat{\chi}_{[N/n]}(1) \right) x^k. \end{aligned}$$

On the other hand, by Theorem 3,

$$\begin{aligned} \frac{\hat{\chi}_N(x)}{1-x} &= \hat{\chi}_N(0) \exp\left(\log \frac{1}{1-x} \right) \exp\left(\sum_{k \geqslant 1} \frac{1}{k} \kappa_{N,k} x^k \right) \\ &= \hat{\chi}_N(0) \exp\left(\sum_{k \geqslant 1} \frac{1}{k} (1 + \kappa_{N,k}) x^k \right) \\ &= \hat{\chi}_N(0) \prod_{k \geqslant 1} \left\{ \sum_{\nu \geqslant 0} \frac{1}{\nu!} \left(\frac{1}{k} (\kappa_{N,k} + 1) \right)^{\nu} x^{k\nu} \right\} \\ &= \hat{\chi}_N(0) \sum_{k \geqslant 0} F_{N,k} x^k \qquad \text{by definition of } F_{N,k}, \end{aligned}$$

and equating coefficients shows the assertions. ∎

In particular, the first equation of Theorem 4 gives for $k \to \infty$:

COROLLARY. $\lim_{k \to \infty} F_{N,k} = \hat{\chi}_N(1)/\hat{\chi}_N(0)$.

REMARK. By definition of the coefficients $F_{N,k}$ we have

$$\begin{aligned}\sum_{k \geqslant 0} F_{N,k} x^k &= \exp\left(\sum_{k \geqslant 1} \frac{1}{k}(1+\kappa_{N,k}) x^k \right) \\ &= \frac{1}{1-x} \exp\left(\sum_{k \geqslant 1} \frac{1}{k} \kappa_{N,k} x^k \right) \\ &= \left(\sum_{k \geqslant 0} x^k \right)\left(\sum_{k \geqslant 0} f_{N,k} x^k \right) \\ &= \sum_{K \geqslant 0} \left(\sum_{k \leqslant K} f_{N,k} \right) x^k\end{aligned}$$

when we define $f_{N,k}$ in the same manner as $F_{N,k}$ but with $\kappa_{N,j}$ instead of $\kappa_{N,j}+1$. Hence

$$F_{N,K} = \sum_{0 \leqslant k \leqslant K} f_{N,k},$$

and, by the corollary,

$$\sum_{k \geqslant 0} f_{N,k} = \frac{\hat{\chi}_N(1)}{\hat{\chi}_N(0)}.$$

2. CONDITIONS ON SMALL EIGENVALUES

THEOREM 5. *If $\varepsilon > 0$ is sufficiently small, and if for every $N \geqslant N_0$ at least one of the polynomials $\chi_M(x)$, $N \leqslant M \leqslant N + N^{1-\varepsilon}$, has a zero λ such that*

$$0.305 \leqslant \lambda \leqslant 1.342,$$

then $\hat{\chi}_N(1) = O(N^{-\varepsilon})$ for $N \to \infty$.

Proof. The interpolation Theorem 2 with $\delta=1$ yields for zeros λ of $\chi_N(x)$

$$\hat{\chi}_N(1)=(\lambda-1)\sum_{2\leqslant n\leqslant N}\frac{\tau(n)}{n(n-\lambda)}\hat{\chi}_{[N/n]}(1), \tag{2.1}$$

and a numerical calculation shows

$$|\lambda-1|\sum_{n\geqslant 2}\frac{\tau(n)}{n(n-\lambda)}<1, \qquad \text{if} \quad 0.304\leqslant\lambda\leqslant 1.343.$$

Hence there exist positive $\varepsilon_1,\varepsilon_2$ such that

$$|\lambda-1|\sum_{n\geqslant 2}\frac{\tau(n)}{n^{1-\varepsilon_1}(n-\lambda)}<1-\varepsilon_2 \qquad \text{if} \quad 0.305\leqslant\lambda\leqslant 1.342. \tag{2.2}$$

We choose $\varepsilon\leqslant\varepsilon_1$, N_0 arbitrary, and N_1 large enough that

$$N_1\geqslant N_0, \tag{2.3}$$

$$\sum_{N<n\leqslant N+N^{1-\varepsilon}}\tau(n)\leqslant 2\zeta_\infty N^{1-\varepsilon} \qquad \text{for} \quad N\geqslant N_1, \tag{2.4}$$

which by the Lemma in the Appendix of [6] is possible, and

$$\left(1+N_1^{-1}\right)^{\varepsilon}\leqslant(1-\varepsilon_2)^{-1}. \tag{2.5}$$

Then we choose C such that

$$C\geqslant 2\zeta_\infty\varepsilon_2^{-1}, \tag{2.6}$$

and

$$C\geqslant(N+1)^{\varepsilon}|\hat{\chi}_N(1)| \qquad \text{for} \quad N=1,2,3,\ldots,N_1. \tag{2.7}$$

Now we prove by induction

$$|\hat{\chi}_N(1)|\leqslant C(N+1)^{-\varepsilon} \qquad \text{for} \quad N\geqslant 1. \tag{2.8}$$

This is true by (2.7) for $N \leqslant N_1$. Suppose now that $N > N_1$ and that (2.8) is true for all $M < N$. Then $N > N_0$ by (2.3), and by assumption there exists a zero $\lambda \in [0.305, 1.342]$ of $\chi_M(x)$ for some $M \in [N, N + N^{1-\varepsilon}]$. We have

$$|\hat{\chi}_M(1)| \leqslant |\lambda - 1| \sum_{2 \leqslant n \leqslant M} \frac{\tau(n)}{n(n-\lambda)} |\hat{\chi}_{[M/n]}(1)|$$

by (2.1), and $[M/2] < N$, since $M \leqslant N + N^{1-\varepsilon} < 2N$. Thus the induction hypothesis gives

$$\begin{aligned} |\hat{\chi}_M(1)| &\leqslant |\lambda - 1| \sum_{2 \leqslant n \leqslant M} \frac{\tau(n)}{n(n-\lambda)} \frac{C}{([M/n]+1)^{\varepsilon}} \\ &\leqslant CM^{-\varepsilon} |\lambda - 1| \sum_{2 \leqslant n \leqslant M} \frac{\tau(n)}{n^{1-\varepsilon}(n-\lambda)} \\ &\leqslant CM^{-\varepsilon}(1 - \varepsilon_2) \qquad \text{by (2.2) and } \varepsilon \leqslant \varepsilon_1. \end{aligned}$$

If $M = N$, then (2.8) follows from this with (2.5) and $N \geqslant N_1$. If $M > N$, then

$$\begin{aligned} |\hat{\chi}_M(1) - \hat{\chi}_N(1)| &= \left| \sum_{N < n \leqslant M} \frac{\rho(n)}{n} \right| && \text{by (0.2)} \\ &\leqslant \sum_{N < n \leqslant M} \frac{\tau(n)}{n} && \text{by definition of } \rho \text{ and } \tau \\ &\leqslant \frac{1}{N+1} \sum_{N < n \leqslant M} \tau(n) \\ &\leqslant 2\zeta_\infty (N+1)^{-\varepsilon} && \text{by (2.4) and } M \leqslant N + N^{1-\varepsilon}. \end{aligned}$$

Hence we get

$$\begin{aligned} |\hat{\chi}_N(1)| &\leqslant |\hat{\chi}_M(1)| + 2\zeta_\infty (N+1)^{-\varepsilon} \\ &\leqslant CM^{-\varepsilon}(1 - \varepsilon_2) + 2\zeta_\infty (N+1)^{-\varepsilon} \\ &\leqslant C(N+1)^{-\varepsilon} \qquad \text{by (2.6)}. \end{aligned}$$

■

REMARK. The proof shows that the admissible λ-interval is determined completely by the inequality

$$|\lambda-1|\sum_{n\geqslant 2}\frac{\tau(n)}{n|n-\lambda|}<1.$$

All induction proofs here will be performed according to this method, and the assumptions of the theorems will be the same in principle. Therefore we formulate them more pointedly—and less precisely—in this way: If sufficiently many polynomials $\chi_N(x)$ have zeros in $[0.305, 1.342]$, then $\hat{\chi}_N(1)=O(N^{-\varepsilon})$.

THEOREM 6. *If sufficiently many polynomials $\chi_N(x)$ have nonreal zeros λ such that*

$$|\lambda-0.426|\leqslant 1.016,$$

then $\hat{\chi}_N(1)=O(N^{-\varepsilon})$.

Proof. Suppose λ to be a nonreal zero of $\chi_N(x)$. Then, by (2.1),

$$\sum_{1\leqslant n\leqslant N}\frac{\tau(n)}{n(n-\lambda)}\hat{\chi}_{[N/n]}(1)=0,$$

and we also have the same equation with $\bar{\lambda}$ instead of λ. The difference of these, multiplied by $|1-\lambda|^2(\lambda-\bar{\lambda})^{-1}$, is

$$-\hat{\chi}_N(1)=|1-\lambda|^2\sum_{2\leqslant n\leqslant N}\frac{\tau(n)}{n|n-\lambda|^2}\hat{\chi}_{[N/n]}(1).$$

Hence the induction proof of Theorem 5 can be copied provided the condition

$$|1-\lambda|^2\sum_{n\geqslant 2}\frac{\tau(n)}{n|n-\lambda|^2}\leqslant 1-\varepsilon_0 \tag{2.9}$$

for a nonreal zero of $\chi_N(x)$ is satisfied for sufficiently many N. Now,

$$|1-\lambda|^2 \sum_{n\geqslant 2} \frac{\tau(n)}{n|n-\lambda|^2} \leqslant \sup_{n\geqslant 2}\left(\left|\frac{1-\lambda}{1-\lambda/n}\right|^2\right) \sum_{n\geqslant 2} \frac{\tau(n)}{n^3},$$

and

$$\sum_{n\geqslant 2} \frac{\tau(n)}{n^3} = \frac{1}{c}, \qquad c = 2.53097.$$

For complex numbers z and $n \geqslant 2$,

$$\left|\frac{1-z}{1-z/n}\right|^2 < c$$

is equivalent to

$$|z-m_n| < r_n$$

with $m_n = (1-c/n)(1-c/n^2)^{-1}$ and $r_n = \sqrt{c}\,(1-1/n)(1-c/n^2)^{-1}$. Hence (2.9) is true if λ lies in the intersection of the circles $K_{r_n-\varepsilon_1}(m_n)$, $n \geqslant 2$, of center m_n and radius $r_n - \varepsilon_1$. The circles $K_{r_n}(m_n)$ cut the real axis at $m_n \pm r_n$, and the sequences

$$(m_n - r_n)_{n\geqslant 2} \quad \text{and} \quad (m_n + r_n)_{n\geqslant 2}$$

increase monotonically, for the functions

$$f_\pm(x) = \left(1-\frac{c}{x}\right)\left(1-\frac{c}{x^2}\right)^{-1} \pm \sqrt{c}\left(1-\frac{1}{x}\right)\left(1-\frac{c}{x^2}\right)^{-1}$$

have the derivatives

$$f'_\pm(x) = (x^2-c)^{-2}(c \pm \sqrt{c}\,)(x \mp \sqrt{c}\,)^2,$$

which are positive for $x \geqslant 2$.

In particular the intersection of the circles $K_{r_n}(m_n)$, $n \geqslant 2$, contains the circle $K_{r_\infty}(m_\infty)$ with

$$m_\infty - r_\infty = \lim_{n\to\infty}(m_n - r_n) = 1 - \sqrt{c} = -0.59090,$$

$$m_\infty + r_\infty = m_2 + r_2 = \frac{4-2c+2\sqrt{c}}{4-c} = 1.44304,$$

and this gives $m_\infty = 0.42607$ and $r_\infty = 1.01697$. ∎

Remark 1. The interpolation Theorem 2 also allows one to derive conditional results involving several zeros of $\chi_N(x)$, e.g.: If sufficiently many polynomials $\chi_N(x)$ have zeros $\lambda_1, \lambda_2, \lambda_3$ such that

$$\left|\prod_{j=1}^{3}(\lambda_j - 1)\right| \sup_{n\geqslant 4}\left|\prod_{j=1}^{3}\frac{n-j}{n-\lambda_j}\right| \leqslant 2.704, \tag{2.10}$$

then $\hat{\chi}_N(1) = O(N^{-\varepsilon})$. For the interpolation theorem with $M = 3$ yields

$$\hat{\chi}_N(1) = \frac{1}{2}\prod_{j=1}^{3}(\lambda_j - 1)\sum_{4\leqslant n\leqslant N}\left(\prod_{j=1}^{3}\frac{n-j}{n-\lambda_j}\right)\frac{\tau(n)}{(n-1)n}\hat{\chi}_{[N/n]}(1),$$

and hence

$$|\hat{\chi}_N(1)| \leqslant \left|\prod_{j=1}^{3}(\lambda_j - 1)\right|\left(\sup_{n\geqslant 4}\left|\prod_{j=1}^{3}\frac{n-j}{n-\lambda_j}\right|\right)\frac{1}{2}$$
$$\times \sum_{4\leqslant n\leqslant N}\frac{\tau(n)}{(n-1)n}|\hat{\chi}_{[N/n]}(1)|,$$

where $\sum_{n\geqslant 4}\tau(n)/(n-1)n = 0.739548$ and therefore

$$2\left(\sum_{n\geqslant 4}\frac{\tau(n)}{n(n-1)}\right)^{-1} > 2.7043.$$

The condition (2.10) is true for instance if $\lambda_1, \lambda_2 \in [0,2]$ and $\lambda_3 \in [-0.8, 2.9]$.

REMARK 2. The starting points in the proofs of Theorems 5 and 6 were equations of the type

$$\sum_{1 \leqslant n \leqslant N} g_n(\lambda_1, \dots, \lambda_r) \hat{\chi}_{[N/n]}(1) = 0 \tag{2.11}$$

with rational functions g_n in the variables $(\lambda_1, \dots, \lambda_r) = \Lambda$. (2.11) can be transformed with Equation (1.1) into

$$\begin{aligned} 0 &= \sum_{1 \leqslant n \leqslant N} g_n(\Lambda) \sum_{1 \leqslant m \leqslant N/n} \frac{\rho(m)}{m^2} \hat{\chi}_{[N/mn]}(0) \\ &= \sum_{1 \leqslant k \leqslant N} \frac{1}{k^2} \left\{ \sum_{n \mid k} n^2 g_n(\Lambda) \rho\left(\frac{k}{n}\right) \right\} \hat{\chi}_{[N/k]}(0). \end{aligned}$$

This shifts the induction proofs to the point $x = 0$.

One example (without proof): If sufficiently many polynomials $\chi_N(x)$ have zeros λ such that $|\lambda| \leqslant 0.409$, then $\hat{\chi}_N(0) = O(N^{-\varepsilon})$.

THEOREM 7. *If sufficiently many polynomials $\chi_N(x)$ have zeros λ such that*

$$-0.09 \leqslant \lambda \leqslant 1.04,$$

then $\hat{\chi}_N(0) = O(N^{-\varepsilon})$ and $\hat{\chi}_N(1) = O(N^{-\varepsilon})$.

REMARK. The two estimates in the conclusion of the theorem are equivalent by (1.1) and (1.2), but with regard to our induction technique it may be worthwhile to note them both, for the following reason: It may be that each single condition of our theorems applies too seldom, i.e. not at least once in every interval $[N, N + N^{1-\varepsilon}]$, $N \geqslant N_0$, as demanded in Theorem 5. Then one can still try to combine these conditions, which means to jump after an induction step to another condition. But then one has to observe the O-constants involved and to distinguish between the points $x = 0$ and $x = 1$. Jumping from estimates of $\hat{\chi}_N(1)$ to those of $\hat{\chi}_N(0)$ or vice versa forces us to enlarge the multiplicative O-constant by the factor ζ_∞ or ρ_∞. But the proof of

Theorem 7 shows that here the induction step is carried out at $x = 0$ and $x = 1$ simultaneously, which means without enlarging either of the O-constants.

However, under appropriate conditions jumping without enlarging the O-constant is possible; e.g., if $\chi_N(x)$ has a zero λ such that $-1.50 \leqslant \lambda \leqslant 0.44$, then $|\hat{\chi}_n(1)| \leqslant Cn^{-\varepsilon}$ for $1 \leqslant n \leqslant N$ implies $|\hat{\chi}_N(0)| \leqslant CN^{-\varepsilon}$.

Proof of Theorem 7. We start with

$$\hat{\chi}_N(1) - \hat{\chi}_N(0) = \sum_{2 \leqslant n \leqslant N} \frac{\mu(n)}{n^2} \hat{\chi}_{[N/n]}(0) - \sum_{2 \leqslant n \leqslant N} \frac{\tau(n)}{n^3} \hat{\chi}_{[N/n]}(1), \tag{2.12}$$

which is Theorem 1(1) for $k = 1$, and we have

$$\alpha := \sum_{n \geqslant 2} |\mu(n)| n^{-2} = 0.519817\ldots,$$

$$\beta := \sum_{n \geqslant 2} \tau(n) n^{-3} = 0.395105\ldots.$$

For the sake of simplicity, we impose the same O-constants on both estimates and prove by induction

$$|\hat{\chi}_N(0)| \leqslant CN^{-\varepsilon} \quad \text{and} \quad |\hat{\chi}_N(1)| \leqslant CN^{-\varepsilon}. \tag{2.13}$$

In the induction step we distinguish four cases:

(1) *$\chi_N(x)$ has exactly one zero in* $[0,1]$. Then $\chi_N(0)$ and $\chi_N(1)$ have different signs, and so have $\hat{\chi}_N(0)$ and $\hat{\chi}_N(1)$. Hence (2.12) shows that $|\hat{\chi}_N(0)|$ as well as $|\hat{\chi}_N(1)|$ can be estimated by

$$\sum_{2 \leqslant n \leqslant N} \frac{|\mu(n)|}{n^2} |\hat{\chi}_{[N/n]}(0)| + \sum_{2 \leqslant n \leqslant N} \frac{\tau(n)}{n^3} |\hat{\chi}_{[N/n]}(1)|,$$

and the induction hypothesis, together with $\alpha + \beta < 0.915 < 1$, gives (2.13).

(2) *$\chi_N(x)$ has at least two zeros λ_1, λ_2 in* $[0,1]$. Then the interpolation Theorem 2 at $x = 0$ gives

$$\hat{\chi}_N(0) = \tfrac{1}{2}\lambda_1\lambda_2 \sum_{3 \leqslant n \leqslant N} \frac{(n-1)(n-2)}{(n-\lambda_1)(n-\lambda_2)} \frac{\tau(n)}{n^2} \hat{\chi}_{[N/n]}(1),$$

which implies

$$|\hat{\chi}_N(0)| \leqslant \frac{1}{2} \sum_{3 \leqslant n \leqslant N} \frac{\tau(n)}{n^2} |\hat{\chi}_{[N/n]}(1)|,$$

and because

$$\sum_{n \geqslant 3} \frac{\tau(n)}{n^2} = \zeta_\infty - 1 - \tfrac{1}{2} < 0.795 < 2,$$

the induction hypothesis $|\hat{\chi}_M(1)| \leqslant CM^{-\varepsilon}$, $1 \leqslant M \leqslant N-1$, yields $|\hat{\chi}_N(0)| \leqslant CN^{-\varepsilon}$. Again the interpolation Theorem 2, now at $x = 1$, gives

$$\hat{\chi}_N(1) = (\lambda_1 - 1)(\lambda_2 - 1) \sum_{3 \leqslant n \leqslant N} \frac{(n-1)(n-2)}{(n-\lambda_1)(n-\lambda_2)} \frac{\tau(n)}{(n-1)n} \hat{\chi}_{[N/n]}(1),$$

which implies

$$|\hat{\chi}_N(1)| \leqslant \sum_{3 \leqslant n \leqslant N} \frac{\tau(n)}{(n-1)n} |\hat{\chi}_{[N/n]}(1)|,$$

and because

$$\sum_{n \geqslant 3} \frac{\tau(n)}{(n-1)n} < 0.99 < 1$$

the induction hypothesis also yields $|\hat{\chi}_N(1)| \leqslant CN^{-\varepsilon}$.

(3) $\lambda = 1 + \delta$, $0 < \delta \leqslant 0.04$, *is a zero of* $\chi_N(x)$. Then the interpolation theorem at $x = 1$ gives

$$\hat{\chi}_N(1) = \delta \sum_{2 \leqslant n \leqslant N} \frac{\tau(n)}{(n-1-\delta)n} \hat{\chi}_{[N/n]}(1),$$

which implies

$$|\hat{\chi}_N(1)| \leqslant \frac{\delta}{1-\delta} \sum_{2 \leqslant n \leqslant N} \frac{\tau(n)}{(n-1)n} |\hat{\chi}_{[N/n]}(1)|,$$

and because

$$\frac{\delta}{1-\delta} \sum_{n \geqslant 2} \frac{\tau(n)}{(n-1)n} \leqslant \frac{0.04}{0.96} \cdot 1.98955 < 0.083 =: \gamma_1$$

the induction hypothesis at $x = 1$ yields

$$|\hat{\chi}_N(1)| \leqslant \gamma_1 C N^{-\varepsilon}, \qquad \gamma_1 < 1. \tag{2.14}$$

But $\alpha + \beta + \gamma_1 < 0.998 < 1$. Thus (2.12), (2.14), and the induction hypothesis at $x = 0$ and $x = 1$ also yield $|\hat{\chi}_N(0)| \leqslant CN^{-\varepsilon}$.

(4) $\lambda = -\delta$, $0 < \delta \leqslant 0.09$, *is a zero of* $\chi_N(x)$. Then the interpolation theorem at $x = 0$ gives

$$\hat{\chi}_N(0) = -\delta \sum_{2 \leqslant n \leqslant N} \frac{n-1}{n+\delta} \frac{\tau(n)}{n^2} \hat{\chi}_{[N/n]}(1),$$

which implies

$$|\hat{\chi}_N(0)| \leqslant \delta \sum_{2 \leqslant n \leqslant N} \left(\frac{\tau(n)}{n^2} - \frac{\tau(n)}{n^3} \right) |\hat{\chi}_{[N/n]}(1)|,$$

and because

$$\delta \sum_{n \geqslant 2} \left(\frac{\tau(n)}{n^2} - \frac{\tau(n)}{n^3} \right) \leqslant 0.09(1.294856 - 0.395105) < 0.081 =: \gamma_0,$$

the induction hypothesis at $x = 1$ yields

$$|\hat{\chi}_N(0)| \leqslant \gamma_0 C N^{-\varepsilon}, \qquad \gamma_0 < 1, \tag{2.15}$$

and again $\alpha + \beta + \gamma_0 < 0.996 < 1$. Thus (2.12), (2.15), and the induction hypothesis at $x = 0$ and $x = 1$ also yield $|\hat{\chi}_N(1)| \leqslant CN^{-\varepsilon}$. ■

3. CONDITIONS ON γ_N

The quotient

$$\gamma_N = \frac{\hat{\chi}_N(0)}{\hat{\chi}_N(1)}$$

has the obvious representations

$$\gamma_N = \frac{\sum_{1 \leqslant n \leqslant N} \frac{\mu(n)}{n}}{\sum_{1 \leqslant n \leqslant N} \frac{\rho(n)}{n}} \qquad \text{[by (0.1) and (0.2)]},$$

which is useful for numerical computations, and

$$\gamma_N = \frac{\prod_{2 \leqslant n \leqslant N} \left(1 - \frac{1}{n}\right)}{\prod_{\lambda} \left(1 - \frac{1}{\lambda}\right)},$$

which shows that in a sense γ_N measures the average deviation of the zeros λ of $\chi_N(x)$ from $2, 3, \ldots, N$. This second equation is a consequence of

$$\frac{\chi_N(0)}{\chi_N(1)} = \prod_{\lambda} \frac{-\lambda}{1 - \lambda} = \prod_{\lambda} \left(1 - \frac{1}{\lambda}\right)^{-1}$$

transcribed to $\hat{\chi}_N(0)$ and $\hat{\chi}_N(1)$.

If Riemann's hypothesis is true, then "on average" the zeros $\lambda_2, \lambda_3, \ldots, \lambda_N$ of $\chi_N(x)$ should be situated near $2 - \frac{1}{2}, 3 - \frac{1}{2}, \ldots, N - \frac{1}{2}$ (cf. the introduction of [5]), which implies

$$\gamma_N \sim \prod_{2 \leqslant n \leqslant N} \frac{1-\dfrac{1}{n}}{1-\dfrac{1}{n-\frac{1}{2}}} \sim 2.$$

The next theorem suggests $\gamma_N \sim \zeta_\infty = 2.294856$ on average. But it seems improbable that the sign changes of the sequence $(\hat{\chi}_N(0))_{N \geqslant 2}$ correspond exactly with those of $(\hat{\chi}_N(1))_{N \geqslant 2}$; hence $(\gamma_N)_{N \geqslant 2}$ should not converge.

THEOREM 8.

(1)
$$\lim_{\varepsilon \downarrow 0} \frac{\displaystyle\sum_{n \geqslant 1} \frac{\mu(n)}{n^{1+\varepsilon}}}{\displaystyle\sum_{n \geqslant 1} \frac{\rho(n)}{n^{1+\varepsilon}}} = \zeta_\infty.$$

(2)
$$\frac{\displaystyle\sum_{N \geqslant 1} \hat{\chi}_N(0) \log\left(1+\frac{1}{N}\right)}{\displaystyle\sum_{N \geqslant 1} \hat{\chi}_N(1) \log\left(1+\frac{1}{N}\right)} = \zeta_\infty.$$

REMARK. $\sum_{N \geqslant 1} \hat{\chi}_N(0) \log(1+1/N) = 1$ is a consequence of the prime number theorem.

Proof of Theorem 8. (1):

$$\lim_{\varepsilon \downarrow 0} \frac{\displaystyle\sum_{n \geqslant 1} \frac{\mu(n)}{n^{1+\varepsilon}}}{\displaystyle\sum_{n \geqslant 1} \frac{\rho(n)}{n^{1+\varepsilon}}} = \lim_{\varepsilon \downarrow 0} \zeta(1+\varepsilon)^{-1} \left(\sum_{n \geqslant 1} \frac{\tau(n)}{n^{1+\varepsilon}} \right) \qquad \text{by } (0.11)$$

$$= \lim_{\varepsilon \downarrow 0} \zeta(1+\varepsilon)^{-1} \prod_{m \geqslant 0} \zeta(1+\varepsilon+m) \qquad \text{by } (0.9)$$

$$= \lim_{\varepsilon \downarrow 0} \prod_{m \geqslant 2} \zeta(\varepsilon+m) = \prod_{m \geqslant 2} \zeta(m) = \zeta_\infty.$$

(2): Abelian summation shows

$$\sum_{1 \leqslant N \leqslant M} \hat{\chi}_N(0) \log\left(1 + \frac{1}{N}\right)$$

$$= \hat{\chi}_M(0) \sum_{1 \leqslant N \leqslant M} \log \frac{N+1}{N} - \sum_{1 \leqslant K < M} \{\hat{\chi}_{K+1}(0) - \hat{\chi}_K(0)\} \sum_{1 \leqslant N \leqslant K} \log \frac{N+1}{N}$$

$$= \hat{\chi}_M(0) \log(M+1) - \sum_{1 \leqslant K < M} \frac{\mu(K+1)}{K+1} \log(K+1),$$

and, with positive constants c_1, c_2, and $c < 1$,

$$\hat{\chi}_M(0) = \sum_{1 \leqslant n \leqslant M} \frac{\mu(n)}{n} = O(\exp(-c_1 \log^c M)), \tag{3.1}$$

$$\sum_{2 \leqslant K \leqslant M} \frac{\mu(K)}{K} \log K = -1 + O(\exp(-c_2 \log^c M)),$$

which both are equivalent to the prime number theorem [7, p. 108]. Hence it follows for $M \to \infty$

$$\sum_{N \geqslant 1} \hat{\chi}_N(0) \log\left(1 + \frac{1}{N}\right) = 1,$$

which proves the remark. Now

$$\sum_{N \geqslant 1} \hat{\chi}_N(0) \log\left(1 + \frac{1}{N}\right) = \sum_{N \geqslant 1} \left(\sum_{1 \leqslant n \leqslant N} \frac{\tau(n)}{n^2} \hat{\chi}_{[N/n]}(1) \right) \log\left(1 + \frac{1}{N}\right)$$

by (1.2), and the series on the right side is absolutely convergent, for (3.1) and (1.1) imply

$$\hat{\chi}_N(1) = O(\exp(-c_3 \log^c N)).$$

Hence

$$\sum_{N\geqslant 1}\hat{\chi}_N(0)\log\left(1+\frac{1}{N}\right)=\sum_{k\geqslant 1}\hat{\chi}_k(1)\left(\sum_{\substack{1\leqslant n\leqslant N\\ [N/n]=k}}\frac{\tau(n)}{n^2}\log\frac{N+1}{N}\right)$$

$$=\sum_{k\geqslant 1}\hat{\chi}_k(1)\sum_{n\geqslant 1}\frac{\tau(n)}{n^2}\left(\sum_{\substack{N\geqslant 1\\ nk\leqslant N<nk+n}}\log\frac{N+1}{N}\right)$$

$$=\sum_{k\geqslant 1}\hat{\chi}_k(1)\sum_{n\geqslant 1}\frac{\tau(n)}{n^2}\log\frac{k+1}{k}$$

$$=\zeta_\infty\sum_{k\geqslant 1}\hat{\chi}_k(1)\log\left(1+\frac{1}{k}\right). \qquad \blacksquare$$

Now we impose conditions on γ_N to prove (0.1) and/or (0.2). First of all, if $\gamma_N<0$, then $\hat{\chi}_N(0)$ and $\hat{\chi}_N(1)$ have different signs, which implies the existence of a zero of $\chi_N(x)$ in $[0,1]$, and if this happens sufficiently often, then $\hat{\chi}_N(0)=O(N^{-\varepsilon})$ and $\hat{\chi}_N(1)=O(N^{-\varepsilon})$ by Theorem 7.

THEOREM 9.

(1) *If sufficiently often*

$$\gamma_N\in\left[\varepsilon_0-\frac{1}{\rho_\infty-2},\frac{1}{\rho_\infty}-\varepsilon_0\right]=[-13.706,0.482],$$

then $\hat{\chi}_N(0)=O(N^{-\varepsilon})$.

(2) *If sufficiently often*

$$\gamma_N\notin[2-\zeta_\infty-\varepsilon_0,\zeta_\infty+\varepsilon_0]=[-0.295,2.295],$$

then $\hat{\chi}_N(1)=O(N^{-\varepsilon})$.

(3) *If sufficiently often*

$$\gamma_N\leqslant 1-\frac{6\zeta_\infty-\pi^2}{2\pi^2-15}-\varepsilon_0=0.177,$$

then $\hat{\chi}_N(0)=O(N^{-\varepsilon})$ *and* $\hat{\chi}_N(1)=O(N^{-\varepsilon})$.

Proof. (1): Equation (1.1) and the definition of γ_N show

$$\left(\frac{1}{\gamma_N}-1\right)\hat{\chi}_N(0)=\sum_{2\leqslant n\leqslant N}\frac{\rho(n)}{n^2}\hat{\chi}_{[N/n]}(0).$$

Therefore the induction step can be carried out as soon as we have

$$\sum_{n\geqslant 2}\frac{|\rho(n)|}{n^2}=\rho_\infty-1\leqslant\left|\frac{1}{\gamma_N}-1\right|-\varepsilon_1,$$

which means $\gamma_N\leqslant 1/\rho_\infty-\varepsilon_2$ if γ_N is positive, and $\gamma_N\geqslant -1/(\rho_\infty-2)+\varepsilon_3$ if γ_N is negative.

(2): Similarly, (1.2) and the definition of γ_N show

$$(\gamma_N-1)\hat{\chi}_N(1)=\sum_{2\leqslant n\leqslant N}\frac{\tau(n)}{n^2}\hat{\chi}_{[N/n]}(1). \tag{3.2}$$

Hence the induction step works as soon as we have

$$\sum_{n\geqslant 2}\frac{\tau(n)}{n^2}=\zeta_\infty-1\leqslant|\gamma_N-1|-\varepsilon_1.$$

(3): As in the proof of Theorem 7, we start with

$$\hat{\chi}_N(1)-\hat{\chi}_N(0)=\sum_{2\leqslant n\leqslant N}\frac{\mu(n)}{n^2}\hat{\chi}_{[N/n]}(0)-\sum_{2\leqslant n\leqslant N}\frac{\tau(n)}{n^3}\hat{\chi}_{[N/n]}(1) \tag{3.3}$$

and

$$\alpha:=\sum_{n\geqslant 2}\frac{|\mu(n)|}{n^2}=\frac{\zeta(2)}{\zeta(4)}-1=\frac{15}{\pi^2}-1=0.519817\ldots,$$

$$\beta:=\sum_{n\geqslant 2}\frac{\tau(n)}{n^3}=\frac{\zeta_\infty}{\zeta(2)}-1=\frac{6\zeta_\infty}{\pi^2}-1=0.395105\ldots.$$

We show simultaneously by induction, for a small $\varepsilon_1>0$ and for sufficiently small $\varepsilon > 0$,

$$|\hat{\chi}_N(0)| \leqslant C(\beta+\varepsilon_1)N^{-\varepsilon} \quad \text{and} \quad |\hat{\chi}_N(1)| \leqslant C(1-\alpha)N^{-\varepsilon}.$$

If $\gamma_N < 0$, then $\hat{\chi}_N(0)$ and $\hat{\chi}_N(1)$ have different signs, and (3.3) shows for $x = 0$ and $x = 1$

$$|\hat{\chi}_N(x)| \leqslant \sum_{2 \leqslant n \leqslant N} \frac{|\mu(n)|}{n^2} |\hat{\chi}_{[N/n]}(0)| + \sum_{2 \leqslant n \leqslant N} \frac{\tau(n)}{n^3} |\hat{\chi}_{[N/n]}(1)|$$

$$\leqslant (\alpha+\varepsilon_2)C(\beta+\varepsilon_1)N^{-\varepsilon} + (\beta+\varepsilon_3)C(1-\alpha)N^{-\varepsilon}$$

(by the induction hypothesis, where $\varepsilon_2 \downarrow 0$ and $\varepsilon_3 \downarrow 0$ when $\varepsilon \downarrow 0$)

$$\leqslant C(\beta+\varepsilon_1)N^{-\varepsilon} \qquad \text{when } \varepsilon_2, \varepsilon_3 \text{ are small enough}$$

$$\leqslant C(1-\alpha)N^{-\varepsilon}.$$

Now we suppose $\gamma_N \geqslant 0$. Then by assumption

$$|\hat{\chi}_N(0)| \leqslant \left(1 - \frac{6\zeta_\infty - \pi^2}{2\pi^2 - 15} - \varepsilon_0\right) |\hat{\chi}_N(1)|$$

$$\leqslant \left(1 - \frac{\beta}{1-\alpha} - \varepsilon_0\right) |\hat{\chi}_N(1)|, \tag{3.4}$$

and with (3.3) we get

$$\left(\frac{\beta}{1-\alpha} + \varepsilon_0\right) |\hat{\chi}_N(1)| \leqslant \sum_{2 \leqslant n \leqslant N} \frac{|\mu(n)|}{n^2} |\hat{\chi}_{[N/n]}(0)|$$

$$+ \sum_{2 \leqslant n \leqslant N} \frac{\tau(n)}{n^3} |\hat{\chi}_{[N/n]}(1)|$$

$$\leqslant C(\beta+\varepsilon_1)N^{-\varepsilon}, \tag{3.5}$$

as above, by the induction hypothesis, which proves the induction step at

$x=1$, provided ε_1 has been chosen small enough. And finally, with (3.4) and (3.5) we also find

$$|\hat{\chi}_N(0)| \leqslant \left(1-\frac{\beta}{1-\alpha}\right)C(1-\alpha+\varepsilon_4)N^{-\varepsilon} \leqslant C\beta N^{-\varepsilon}. \qquad \blacksquare$$

REMARK. The upper bound 0.177 for γ_N in Theorem 9(3) can be improved by adding zero conditions, e.g.: If for sufficiently many N, $\chi_N(x)$ has an odd number of zeros in $]1,2[$ and $\gamma_N \leqslant 0.205$, then $\hat{\chi}_N(1) = O(N^{-\varepsilon})$.

The proof starts with Equation (3.2) and makes use of the fact that the zero condition implies $\operatorname{sgn}\chi_N(1) \neq \operatorname{sgn}\chi_N(2)$. Thus, by (0.6) for $n=2$, $\operatorname{sgn}\hat{\chi}_N(1) = \operatorname{sgn}\hat{\chi}_{[N/2]}(1)$, and consequently $(\gamma_N - 1)\hat{\chi}_N(1)$ and $\hat{\chi}_{[N/2]}(1)$ have different signs, provided $\gamma_N < 1$. Hence the induction step works as soon as

$$1-\gamma_N \geqslant \varepsilon_1 + \sum_{n \geqslant 3} \frac{\tau(n)}{n^2} = \varepsilon_1 + 0.794856.$$

If γ_N comes near ζ_∞ too often (as is suggested by Theorem 8), then Theorem 9 is not applicable to prove (0.1) or (0.2). Then we add another zero condition, for arbitrarily large M:

THEOREM 10. *If for a sufficiently dense subsequence* $(N_i)_{i \geqslant 1}$ *of the natural numbers we have*

$$\lim_{i \to \infty} \gamma_{N_i} = \zeta_\infty,$$

and if each of the polynomials $\chi_{N_i}(x)$, $i \geqslant 1$, *has no zeros in at least one of the intervals* $]m, m+1[$, $1 \leqslant m \leqslant M$, *then* $\hat{\chi}_N(1) = O(N^{-\varepsilon})$.

Proof. Equation (0.6) states

$$\frac{(-1)^N}{(N-n)!}\chi_N(n) = (-1)^n n! \frac{\tau(n)}{n^2}\hat{\chi}_{[N/n]}(1) \qquad \text{for} \quad 1 \leqslant n \leqslant N.$$

Hence, with

$$\operatorname{sgn} x := \begin{cases} 1 & \text{if} \quad x > 0, \\ 0 & \text{if} \quad x = 0, \\ -1 & \text{if} \quad x < 0, \end{cases}$$

we find

$$\operatorname{sgn} \hat{\chi}_{[N/n]}(1) = \operatorname{sgn} \hat{\chi}_{[N/(n+1)]}(1)$$

if and only if

$$-\operatorname{sgn} \chi_N(n) = \operatorname{sgn} \chi_N(n+1).$$

If $\chi_N(x)$ has no zero in $]m, m+1[$, we therefore have

$$-\operatorname{sgn} \chi_N(m) \neq \operatorname{sgn} \chi_N(m+1) \quad \text{or} \quad \chi_N(m) = \chi_N(m+1) = 0,$$

and this means

$$\operatorname{sgn} \hat{\chi}_{[N/m]}(1) \neq \operatorname{sgn} \hat{\chi}_{[N/(m+1)]}(1) \quad \text{or} \quad \hat{\chi}_{[N/m]}(1) = \hat{\chi}_{[N/(m+1)]}(1) = 0.$$

Now Equation (3.2) yields

$$|\gamma_N - 1| |\hat{\chi}_N(1)| \leqslant \sum_{2 \leqslant n \leqslant N,\, n \neq c} \frac{\tau(n)}{n^2} |\hat{\chi}_{[N/n]}(1)|$$

with $c = m$ or $c = m+1$.

But $\sum_{n \geqslant 2,\, n \neq c} \tau(n)/n^2 \leqslant \zeta_\infty - 1 - (M+1)^{-2}$, and then the assumption $\lim_{i \to \infty} \gamma_{N_i} = \zeta_\infty$ shows that induction is possible. ∎

REMARK 1. The proof shows that it is sufficient to assume that the polynomials $\chi_{N_i}(x)$ contain an even number of zeros in at least one of the intervals $]m, m+1[$, $m \leqslant M$. But two zeros of $\chi_N(x)$ in the same interval $]m, m+1[$ occur fairly seldom—this seems to be the degenerate case of a pair of complex zeros $\alpha \pm \beta i$ with $\alpha \sim m$, as a comparison of such zeros of $\chi_N(x)$ with the zeros of $\chi_{N \pm 1}(x), \chi_{N \pm 2}(x), \ldots$ suggests. On the other hand there are reasons to suspect that for some constant $c_0 > 0$ the number of intervals $]m, m+1[$, $m \leqslant \sqrt{N}$, without zeros of $\chi_N(x)$ is greater than $c_0\sqrt{N}$.

REMARK 2. Combinations of equations in Theorem 1, for instance

$$\hat{\chi}_N(1) - \hat{\chi}_N(0) = \sum_{2 \leqslant n \leqslant N} \frac{\rho(n)}{n^3} \hat{\chi}_{[N/n]}(0) - \sum_{2 \leqslant n \leqslant N} \frac{1}{n^2} \hat{\chi}_{[N/n]}(1),$$

$$\hat{\chi}_N(1) - \hat{\chi}_N(0) = \sum_{2 \leqslant n \leqslant N} \frac{\mu^{(2)}(n)}{n^2} \hat{\chi}_{[N/n]}(0) - \sum_{2 \leqslant n \leqslant N} \frac{\tau(n)}{n^4} \hat{\chi}_{[N/n]}(1),$$

can be used to prove by induction

$$|\hat{\chi}_N(0)| \leqslant CN^{-\varepsilon} \quad \text{and} \quad |\hat{\chi}_N(1)| \leqslant c_1 CN^{-\varepsilon},$$

if sufficiently often

$$\gamma_N \leqslant 0.177 \qquad (c_1 = 1.215328)$$

or

$$-29.89 \leqslant \gamma_N \leqslant 0.205 \qquad (c_1 = 1.3)$$

or

$$\gamma_N \notin \,]0, 7.110] \qquad (c_1 = 0.859352).$$

4. CONDITIONS ON κ_N, σ_N, AND ν_N

Here we give conditional proofs for (0.1) and (0.2) depending on the parameters

$$\kappa_N = \sum_{2 \leqslant n \leqslant N} \frac{1}{n} - \sum_{\lambda} \frac{1}{\lambda} \left(= \kappa_{N,1}; \text{ cf. Theorem 3} \right),$$

$$\sigma_N = \sum_{2 \leqslant n \leqslant N} \frac{1}{n-1} - \sum_{\lambda} \frac{1}{\lambda - 1},$$

and (cf. Theorem 4)

$$\nu_N = \frac{F_{N,2}}{F_{N,1}} = \frac{1}{2}\left(K_{N,1} + \frac{K_{N,2}}{K_{N,1}}\right)$$

with

$$K_{N,k} = \sum_{1 \leqslant n \leqslant N} n^{-k} - \sum_{\lambda} \lambda^{-k}.$$

As usual, λ runs through the eigenvalues of A_N. κ_N and σ_N are the quotients

$$\kappa_N = \frac{\hat{\chi}_N'(0)}{\hat{\chi}_N(0)} \quad \text{and} \quad \sigma_N = \frac{\hat{\chi}_N'(1)}{\hat{\chi}_N(1)}, \tag{4.1}$$

for logarithmic differentiation of $\hat{\chi}_N(x) = \prod_{2 \leqslant n \leqslant N}(x - \lambda_n)/(x - n)$ yields

$$\frac{\hat{\chi}_N'(x)}{\hat{\chi}_N(x)} = \sum_{2 \leqslant n \leqslant N} \frac{d}{dx}\left(\frac{x - \lambda_n}{x - n}\right)\frac{x - n}{x - \lambda_n}$$

$$= \sum_{2 \leqslant n \leqslant N} \frac{1}{n - x} - \sum_{\lambda} \frac{1}{\lambda - x}.$$

Similar to Theorem 8(2), we have the following average result for κ_N and σ_N:

THEOREM 11.

(1) $$\frac{\sum_{N \geqslant 1} \kappa_N \hat{\chi}_N(0) \log\left(1 + \frac{1}{N}\right)}{\sum_{N \geqslant 1} \hat{\chi}_N(0) \log\left(1 + \frac{1}{N}\right)} = \frac{1}{\zeta(2)} - 1 = -0.392073\ldots,$$

(2) $$\frac{\sum_{N \geqslant 1} \sigma_N \hat{\chi}_N(1) \log\left(1 + \frac{1}{N}\right)}{\sum_{N \geqslant 1} \hat{\chi}_N(1) \log\left(1 + \frac{1}{N}\right)} = -\sum_{n \geqslant 2} \frac{\tau(n)}{(n-1)n} = -1.989548\ldots.$$

Proof. (1): Theorems 1 and 4 show

$$\sum_{1\leqslant n\leqslant N}\frac{\tau(n)}{n^3}\hat{\chi}_{[N/n]}(1)=\sum_{1\leqslant n\leqslant N}\frac{\mu(n)}{n^2}\hat{\chi}_{[N/n]}(0),$$

$$\sum_{1\leqslant n\leqslant N}\frac{\tau(n)}{n^3}\hat{\chi}_{[N/n]}(1)=F_{N,1}\hat{\chi}_N(0),$$

and $F_{N,1}=\kappa_N+1$; hence

$$\kappa_N\hat{\chi}_N(0)=\sum_{2\leqslant n\leqslant N}\frac{\mu(n)}{n^2}\hat{\chi}_{[N/n]}(0). \tag{4.2}$$

As we have seen in the proof of Theorem 8(2), $\Sigma_{k\geqslant 1}\hat{\chi}_k(0)\log(1+1/k)$ converges absolutely, and thus

$$\begin{aligned}
&\frac{1}{\zeta(2)}\sum_{k\geqslant 1}\hat{\chi}_k(0)\log\left(1+\frac{1}{k}\right)\\
&=\sum_{k\geqslant 1}\hat{\chi}_k(0)\left(\sum_{n\geqslant 1}\frac{\mu(n)}{n^2}\log\frac{k+1}{k}\right)\\
&=\sum_{k\geqslant 1}\hat{\chi}_k(0)\left\{\sum_{n\geqslant 1}\frac{\mu(n)}{n^2}\left(\sum_{nk\leqslant N<nk+n}\log\frac{N+1}{N}\right)\right\}\\
&=\sum_{k\geqslant 1}\hat{\chi}_k(0)\sum_{N\geqslant 1}\sum_{\substack{1\leqslant n\leqslant N\\ [N/n]=k}}\frac{\mu(n)}{n^2}\log\frac{N+1}{N}\\
&=\sum_{N\geqslant 1}\left(\sum_{1\leqslant n\leqslant N}\frac{\mu(n)}{n^2}\hat{\chi}_{[N/n]}(0)\right)\log\left(1+\frac{1}{N}\right)\\
&=\sum_{N\geqslant 1}(\kappa_N+1)\hat{\chi}_N(0)\log\left(1+\frac{1}{N}\right)\qquad\text{by (4.2)}\\
&=\sum_{N\geqslant 1}\kappa_N\hat{\chi}_N(0)\log\left(1+\frac{1}{N}\right)+\sum_{N\geqslant 1}\hat{\chi}_N(0)\log\left(1+\frac{1}{N}\right).
\end{aligned}$$

Hence

$$\sum_{N\geqslant 1}\kappa_N\hat{\chi}_N(0)\log\left(1+\frac{1}{N}\right)=\left(\frac{1}{\zeta(2)}-1\right)\sum_{N\geqslant 1}\hat{\chi}_N(0)\log\left(1+\frac{1}{N}\right),$$

and the remark to Theorem 8 ensures that

$$\sum_{N\geqslant 1}\hat{\chi}_N(0)\log\left(1+\frac{1}{N}\right)\neq 0.$$

(2): Equation (0.7) shows

$$\frac{\hat{\chi}_N(x)-\hat{\chi}_N(1)}{x-1}=-\sum_{2\leqslant n\leqslant N}\frac{\tau(n)}{n(n-x)}\hat{\chi}_{[N/n]}(1),$$

and with (4.1) and for $x\to 1$ this yields

$$\sigma_N\hat{\chi}_N(1)=-\sum_{2\leqslant n\leqslant N}\frac{\tau(n)}{(n-1)n}\hat{\chi}_{[N/n]}(1). \tag{4.3}$$

Finally, again

$$\begin{aligned}
&\sum_{k\geqslant 1}\hat{\chi}_k(1)\log\left(1+\frac{1}{k}\right)\left(\sum_{n\geqslant 2}\frac{\tau(n)}{(n-1)n}\right)\\
&\quad=\sum_{k\geqslant 1}\hat{\chi}_k(1)\sum_{n\geqslant 2}\frac{\tau(n)}{(n-1)n}\left(\sum_{nk\leqslant N<nk+n}\log\frac{N+1}{N}\right)\\
&\quad=\sum_{N\geqslant 1}\left(\sum_{2\leqslant n\leqslant N}\frac{\tau(n)}{(n-1)n}\hat{\chi}_{[N/n]}(1)\right)\log\left(1+\frac{1}{N}\right)\\
&\quad=-\sum_{N\geqslant 1}\sigma_N\hat{\chi}_N(1)\log\left(1+\frac{1}{N}\right)\qquad\text{by (4.3)},
\end{aligned}$$

and $\sum_{k\geqslant 1}\hat{\chi}_k(1)\log(1+1/k)\neq 0$ by Theorem 8(2). ∎

COROLLARY.

(1) *If sufficiently often*

$$|\kappa_N| \geqslant \frac{\zeta(2)}{\zeta(4)} - 1 + \varepsilon_0 = 0.520,$$

then $\hat{\chi}_N(0) = O(N^{-\varepsilon})$.

(2) *If sufficiently often*

$$|\sigma_N| \geqslant \sum_{n \geqslant 2} \frac{\tau(n)}{(n-1)n} + \varepsilon_0 = 1.990,$$

then $\hat{\chi}_N(1) = O(N^{-\varepsilon})$.

Proof. This is an immediate consequence of Equations (4.2) and (4.3). ∎

THEOREM 12. *If sufficiently often*

$$\kappa_N \in \left[\varepsilon_0 - \left(1 - \frac{1}{\zeta(2)}\right)\left(1 - \frac{2}{\zeta_\infty}\right)^{-1}, \frac{1}{\zeta(2)} - 1 - \varepsilon_0\right] = [-3.051, -0.393],$$

then $\hat{\chi}_N(1) = O(N^{-\varepsilon})$.

Proof. The first three equations in the proof of Theorem 11 show

$$(\kappa_N + 1)\hat{\chi}_N(0) = \sum_{1 \leqslant n \leqslant N} \frac{\tau(n)}{n^3} \hat{\chi}_{[N/n]}(1),$$

and this combined with Equation (1.2) gives

$$-\kappa_N \hat{\chi}_N(1) = \sum_{2 \leqslant n \leqslant N} \left(\frac{\kappa_N + 1}{n^2} - \frac{1}{n^3}\right) \tau(n) \hat{\chi}_{[N/n]}(1). \tag{4.4}$$

Hence we need

$$|\kappa_N| \geqslant \sum_{n \geqslant 2} \left| \frac{\kappa_N + 1}{n^2} - \frac{1}{n^3} \right| \tau(n) + \varepsilon_1 \tag{4.5}$$

for induction at $x = 1$. We distinguish five cases:

(1) $\kappa_N \leqslant -1$. Then we must have

$$-\kappa_N \geqslant \sum_{n \geqslant 2} \left(-\frac{\kappa_N + 1}{n^2} + \frac{1}{n^3} \right) \tau(n) + \varepsilon_1,$$

which leads to $\kappa_N \geqslant \varepsilon_2 - \{1 - 1/\zeta(2)\}(1 - 2/\zeta_\infty)^{-1}$.
(2) $-1 < \kappa_N \leqslant -\frac{3}{4}$. Then $|\kappa_N| \geqslant \frac{3}{4}$, and

$$\sum_{n \geqslant 2} \left| \frac{\kappa_N + 1}{n^2} - \frac{1}{n^3} \right| \tau(n) \leqslant \frac{1}{4} \sum_{n \geqslant 2} \frac{\tau(n)}{n^2} + \sum_{n \geqslant 2} \frac{\tau(n)}{n^3} < 0.72.$$

(3) $-\frac{3}{4} < \kappa_N \leqslant -\frac{1}{2}$. Then $|\kappa_N| \geqslant \frac{1}{2}$, and

$$\sum_{n \geqslant 2} \left| \frac{\kappa_N + 1}{n^2} - \frac{1}{n^3} \right| \tau(n)$$

$$\leqslant 2 \times \frac{1}{16} + \frac{3}{2} \times \frac{1}{54} + \frac{1}{2} \sum_{n \geqslant 4} \frac{\tau(n)}{n^2} - \sum_{n \geqslant 4} \frac{\tau(n)}{n^3} \leqslant 0.38.$$

(4) $-\frac{1}{2} < \kappa_N \leqslant 0$. Then we must have

$$-\kappa_N \geqslant \sum_{n \geqslant 2} \left(\frac{\kappa_N + 1}{n^2} - \frac{1}{n^3} \right) \tau(n) + \varepsilon_1,$$

which leads to $\kappa_N \leqslant 1/\zeta(2) - 1 - \varepsilon_2$.
(5) $0 < \kappa_N$. Then (4.5) has no solution. ∎

THEOREM 13. *If sufficiently often*

$$\nu_N \notin \left[\frac{1}{\zeta(3)} - \varepsilon_0, \left(2 - \frac{\zeta_\infty}{\zeta(2)\zeta(3)}\right)\left(2 - \frac{\zeta_\infty}{\zeta(2)}\right)^{-1} + \varepsilon_0\right] = [0.831, 1.388],$$

then $\hat{\chi}_N(1) = O(N^{-\varepsilon})$.

Proof. By Theorem 4 we have

$$F_{N,1}\hat{\chi}_N(0) = \sum_{1 \leqslant n \leqslant N} \frac{\tau(n)}{n^3}\hat{\chi}_{[N/n]}(1),$$

$$F_{N,2}\hat{\chi}_N(0) = \sum_{1 \leqslant n \leqslant N} \frac{\tau(n)}{n^4}\hat{\chi}_{[N/n]}(1).$$

Hence, with $\nu_N = F_{N,2}/F_{N,1}$,

$$\nu_N \sum_{1 \leqslant n \leqslant N} \frac{\tau(n)}{n^3}\hat{\chi}_{[N/n]}(1) = \sum_{1 \leqslant n \leqslant N} \frac{\tau(n)}{n^4}\hat{\chi}_{[N/n]}(1),$$

and, as starting point for the induction proof at $x = 1$,

$$(1 - \nu_N)\hat{\chi}_N(1) = \sum_{2 \leqslant n \leqslant N} \left(\frac{\nu_N}{n^3} - \frac{1}{n^4}\right)\tau(n)\hat{\chi}_{[N/n]}(1).$$

So we must have

$$|1 - \nu_N| \geqslant \sum_{n \geqslant 2} \left|\frac{\nu_N}{n^3} - \frac{1}{n^4}\right|\tau(n) + \varepsilon_1,$$

which is certainly true for $\nu_N \leqslant \frac{1}{2}$.
If $\frac{1}{2} < \nu_N \leqslant 1$, we must have

$$1 - \nu_N \geqslant \nu_N \sum_{n \geqslant 2} \frac{\tau(n)}{n^3} - \sum_{n \geqslant 2} \frac{\tau(n)}{n^4} + \varepsilon_1,$$

i.e.

$$\sum_{n\geqslant 1}\frac{\tau(n)}{n^4}-\varepsilon_1\geqslant \nu_N\sum_{n\geqslant 1}\frac{\tau(n)}{n^3},$$

which with (0.9) gives the lower bound $1/\zeta(3)-\varepsilon_0$ of the ν_N-interval. Finally if $\nu_N>1$, we must have

$$\nu_N-1\geqslant \nu_N\sum_{n\geqslant 2}\frac{\tau(n)}{n^3}-\sum_{n\geqslant 2}\frac{\tau(n)}{n^4}+\varepsilon_1,$$

i.e.

$$\nu_N\left(2-\sum_{n\geqslant 1}\frac{\tau(n)}{n^3}\right)\geqslant 2-\sum_{n\geqslant 1}\frac{\tau(n)}{n^4}+\varepsilon_1,$$

which gives the upper bound $(2-\zeta_\infty/\zeta(2)\zeta(3))(2-\zeta_\infty/\zeta(2))^{-1}+\varepsilon_0$ of the ν_N-interval. ∎

Remark. In the same way induction proofs can be established with systems of parameters and systems of equations. For instance, the equations

$$(\gamma_N-1)\hat{\chi}_N(1)=\sum_{2\leqslant n\leqslant N}\frac{\tau(n)}{n^2}\hat{\chi}_{[N/n]}(1),$$

$$\sigma_N\hat{\chi}_N(1)=-\sum_{2\leqslant n\leqslant N}\frac{\tau(n)}{(n-1)n}\hat{\chi}_{[N/n]}(1)$$

yield a result of the following form: $\hat{\chi}_N(1)=O(N^{-\varepsilon})$ is true if sufficiently often the pair (γ_N,σ_N) lies outside a lens-shaped region inside the rectangle

$$|\gamma_N|\leqslant\zeta_\infty+\varepsilon_0,\qquad |\sigma_N|\leqslant\tau_\infty+\varepsilon_0.$$

APPENDIX. NUMERICAL EXAMPLES

TABLE 1[a]

N	λ_-	$\lambda_+^{(1)}$	$\lambda_+^{(2)}$	$\lambda_+^{(3)}$	λ_c	A_-	A_c
101	—	2.442	7.004	9.332	$-2.333 \pm 2.717i$	0	10
102	—	2.432	7.006	9.303	$-1.076 \pm 3.385i$	0	10
103	-0.496	2.514	7.006	9.298	$3.666 \pm 1.144i$	2	8
104	-0.627	2.879	7.003	9.232	$2.596 \pm 3.140i$	2	8
105	-1.969	1.255	3.084	6.812	$5.274 \pm 0.254i$	1	8
106	—	3.079	6.811	9.315	$-0.074 \pm 0.769i$	0	10
107	-3.624	0.879	3.076	6.808	$5.262 \pm 0.276i$	1	8
108	-3.689	0.903	3.166	5.190	$2.707 \pm 4.245i$	1	6
109	-5.991	1.165	3.161	5.197	$3.263 \pm 3.951i$	1	6
110	-5.001	1.648	3.159	4.880	$2.888 \pm 5.482i$	1	6
111	-4.429	1.504	3.057	4.876	$2.485 \pm 5.120i$	1	6
112	-4.173	1.562	3.083	4.739	$3.418 \pm 5.653i$	1	6
113	-6.168	1.636	3.078	4.729	$3.985 \pm 5.234i$	1	6
114	-5.340	2.035	3.410	4.848	$6.500 \pm 0.682i$	1	8
115	-4.724	2.040	3.451	4.999	$2.346 \pm 5.396i$	1	8
116	-3.994	2.402	3.739	4.998	$1.607 \pm 5.421i$	1	8
117	-3.142	2.315	3.809	4.999	$0.850 \pm 6.291i$	1	8
118	-2.388	2.213	3.803	4.999	$0.360 \pm 6.125i$	1	8
119	-1.074	2.235	3.817	4.999	$-0.942 \pm 6.316i$	1	8
120	-1.295	2.577	7.114	9.613	$-0.440 \pm 4.372i$	1	10
121	-1.592	2.580	7.117	9.575	$-0.858 \pm 3.870i$	1	10
122	-0.115	2.537	7.117	9.576	$-1.758 \pm 4.160i$	1	10
123	—	0.687	2.630	7.116	$4.866 \pm 0.187i$	0	10
124	—	0.496	2.733	7.113	$4.936 \pm 0.206i$	0	10
125	—	0.495	2.732	7.115	$4.929 \pm 0.250i$	0	10
126	—	0.689	2.954	7.003	$-1.736 \pm 4.703i$	0	10
127	-0.146	2.953	7.003	9.007	$-1.981 \pm 2.200i$	1	10
128	-0.146	2.953	7.003	9.007	$-1.981 \pm 2.201i$	1	10
129	—	0.808	3.032	7.003	$-2.784 \pm 3.436i$	0	10
130	-0.060	3.032	7.003	9.005	$-1.415 \pm 4.603i$	1	10
131	-3.511	3.033	7.003	9.005	$-0.348 \pm 2.404i$	1	10
132	-4.188	3.286	5.354	6.821	$1.414 \pm 1.655i$	1	8
133	-0.491	3.266	5.388	9.003	$-0.765 \pm 1.809i$	1	10
134	—	1.375	3.280	5.384	$-2.033 \pm 3.257i$	0	10
135	—	1.414	3.361	5.341	$-1.334 \pm 3.356i$	0	10
136	—	1.850	3.768	5.491	$-0.020 \pm 3.870i$	0	10
137	-3.268	0.216	1.787	3.777	$3.901 \pm 4.608i$	1	8
138	-2.438	3.347	5.993	8.686	$1.615 \pm 5.030i$	1	10
139	-5.540	3.395	5.993	8.682	$3.045 \pm 1.090i$	1	10
140	-5.615	1.540	2.844	4.346	$6.298 \pm 1.027i$	1	8
141	-5.000	1.139	2.582	4.314	$6.239 \pm 1.007i$	1	8

TABLE 1[a] *Continued*

N	λ_-	$\lambda_+^{(1)}$	$\lambda_+^{(2)}$	$\lambda_+^{(3)}$	λ_c	A_-	A_c
142	−4.292	0.136	2.665	4.302	$6.210 \pm 0.993i$	1	8
143	—	2.630	4.333	5.991	$-1.904 \pm 1.819i$	0	10
144	—	2.679	4.557	5.734	$-2.023 \pm 1.790i$	0	10
145	—	2.633	4.757	5.663	$3.409 \pm 2.435i$	0	10
146	—	2.690	4.749	5.657	$3.673 \pm 2.331i$	0	10
147	—	2.725	4.714	5.567	$3.213 \pm 2.711i$	0	10
148	—	2.391	4.425	5.485	$3.701 \pm 1.724i$	0	10
149	—	2.428	4.393	5.462	$3.865 \pm 2.040i$	0	10
150	—	2.398	8.014	9.512	$-2.719 \pm 2.827i$	0	12
151	−1.143	2.424	8.014	9.508	$4.145 \pm 0.448i$	2	10
152	−1.334	2.688	7.366	9.460	$5.498 \pm 1.304i$	2	10
153	—	2.902	7.282	9.300	$-2.447 \pm 1.545i$	0	12
154	−0.126	2.892	6.750	9.315	$5.269 \pm 0.865i$	2	10
155	—	2.872	6.727	9.311	$1.503 \pm 2.462i$	0	12
156	—	2.839	6.902	9.467	$-0.842 \pm 1.722i$	0	12
157	−0.031	2.843	6.901	9.462	$4.929 \pm 0.618i$	2	10
158	−1.495	2.852	6.901	9.461	$4.920 \pm 0.619i$	2	10
159	—	2.779	6.900	9.460	$-2.214 \pm 2.475i$	0	12
160	—	2.786	6.925	9.268	$-2.175 \pm 2.419i$	0	12
161	—	2.776	6.999	9.258	$-2.266 \pm 4.367i$	0	12
162	—	2.783	6.999	9.005	$-2.125 \pm 4.208i$	0	12
163	—	2.788	6.999	9.005	$-3.326 \pm 1.053i$	0	12
164	—	2.708	6.999	9.005	$-3.035 \pm 2.726i$	0	12
165	—	2.847	6.999	9.005	$-0.770 \pm 2.347i$	0	12
166	—	2.857	6.999	9.005	$-0.510 \pm 4.326i$	0	12
167	−2.025	2.861	6.999	9.005	$5.014 \pm 0.502i$	2	10
168	−1.337	2.730	6.867	9.006	$4.313 \pm 0.419i$	2	10
169	−1.157	2.726	6.872	9.006	$4.305 \pm 0.433i$	2	10
170	−0.251	2.685	4.453	4.719	$4.992 \pm 5.261i$	2	8
171	−0.601	3.059	6.922	8.654	$4.684 \pm 0.283i$	2	10
172	—	1.662	2.394	4.789	$0.248 \pm 3.627i$	0	10
173	−5.716	2.047	4.802	6.917	$3.024 \pm 0.618i$	1	10
174	−4.362	2.418	4.303	6.881	$5.206 \pm 0.718i$	1	10
175	−3.031	2.385	5.430	6.855	$4.322 \pm 0.449i$	1	10
176	−2.851	2.514	5.146	6.547	$3.910 \pm 1.579i$	1	10
177	−1.394	2.707	5.139	6.537	$3.801 \pm 1.928i$	1	10
178	—	0.234	2.623	5.136	$3.706 \pm 2.182i$	0	10
179	−3.626	2.582	5.131	6.518	$3.086 \pm 2.186i$	1	10
180	−5.077	2.559	7.104	10.007	$-0.022 \pm 2.154i$	1	12
181	−8.529	2.543	7.102	10.007	$0.992 \pm 2.169i$	1	12
182	−6.991	2.596	7.369	10.008	$1.374 \pm 3.234i$	1	12
183	−6.316	2.696	7.366	10.008	$0.653 \pm 2.858i$	1	12
184	−5.295	2.507	4.278	4.728	$0.711 \pm 3.762i$	1	10

TABLE 1[a] *Continued*

N	λ_-	$\lambda_+^{(1)}$	$\lambda_+^{(2)}$	$\lambda_+^{(3)}$	λ_c	A_-	A_c
185	-2.892	2.548	4.200	4.931	$-1.125 \pm 3.300i$	1	10
186	-2.313	2.508	4.125	4.956	$-1.360 \pm 5.422i$	1	10
187	-0.718	2.525	4.119	4.958	$-2.893 \pm 6.385i$	1	10
188	-0.521	2.642	4.297	4.951	$7.560 \pm 0.559i$	1	10
189	-0.515	2.673	4.326	4.947	$8.105 \pm 0.082i$	1	10
190	-0.896	2.689	7.985	9.579	$4.549 \pm 0.034i$	1	12
191	-1.966	2.683	7.984	9.586	$4.551 \pm 0.080i$	1	12
192	-1.965	2.683	7.483	9.487	$4.560 \pm 0.131i$	1	12
193	-4.661	2.677	7.473	9.495	$4.562 \pm 0.152i$	1	12
194	-3.505	2.656	7.471	9.496	$4.565 \pm 0.165i$	1	12
195	-3.018	2.565	7.528	9.471	$4.477 \pm 0.304i$	1	12
196	-2.966	2.549	7.239	9.573	$4.465 \pm 0.267i$	1	12
197	-5.515	2.540	7.234	9.580	$4.465 \pm 0.281i$	1	12
198	-5.784	2.684	7.273	8.541	$4.408 \pm 0.693i$	1	12
199	-8.261	2.673	7.267	8.530	$4.405 \pm 0.727i$	1	12
200	-8.149	2.703	7.263	8.740	$4.492 \pm 0.844i$	1	12

[a] λ_- denotes the largest negative zero of $\chi_N(x)$; $\lambda_+^{(1)}$, $\lambda_+^{(2)}$, $\lambda_+^{(3)}$ are the smallest three positive zeros of $\chi_N(x)$; λ_c denotes the complex zero of $\chi_N(x)$ with minimal modulus; A_- and A_c count the number of negative and of complex zeros of $\chi_N(x)$.

TABLE 2

N	γ_N	κ_N	σ_N	ν_N
101	0.412	0.733	1.124	1.219
102	0.563	0.403	0.819	1.155
103	0.164	2.622	1.576	1.398
104	0.119	2.622	2.097	1.607
105	1.226	0.434	-2.226	0.879
106	0.127	0.981	2.612	2.165
107	-2.251	-0.196	9.773	0.395
108	-2.938	-0.196	11.781	0.443
109	2.547	-0.084	-4.654	0.776
110	0.821	0.282	-0.042	1.011
111	0.957	0.236	-0.480	0.968
112	0.929	0.236	-0.335	0.968
113	0.949	0.158	-0.222	0.978
114	0.629	0.398	0.557	1.088
115	0.594	0.447	0.627	1.104
116	0.467	0.620	0.993	1.173

TABLE 2 *Continued*

N	γ_N	κ_N	σ_N	ν_N
117	0.426	0.738	1.048	1.192
118	0.401	0.841	1.063	1.202
119	0.273	1.429	1.329	1.285
120	0.231	1.429	1.661	1.371
121	0.236	1.346	1.673	1.380
122	0.038	9.507	2.187	1.688
123	−0.810	−0.667	4.501	−1.066
124	−0.363	−1.237	3.305	5.722
125	−0.361	−1.237	3.301	5.759
126	−0.745	−0.616	4.669	−0.774
127	0.032	8.012	2.584	1.971
128	0.032	8.012	2.584	1.971
129	−1.231	−0.242	6.803	0.249
130	0.019	17.576	2.406	1.767
131	0.237	1.107	1.869	1.423
132	0.460	0.430	1.351	1.192
133	0.074	3.166	2.599	1.790
134	1.018	0.281	−0.989	0.982
135	0.934	0.281	−0.689	1.021
136	0.664	0.281	0.515	1.132
137	−0.157	−4.055	1.784	1.822
138	0.390	0.780	1.246	1.209
139	0.551	0.407	0.914	1.137
140	1.134	0.078	−0.619	0.922
141	3.437	−0.180	−6.045	0.679
142	−0.064	−6.615	2.341	1.893
143	0.306	1.011	1.454	1.316
144	0.303	1.011	1.484	1.316
145	0.445	0.607	1.119	1.202
146	0.461	0.577	1.083	1.190
147	0.480	0.512	1.073	1.187
148	0.617	0.339	0.712	1.114
149	0.518	0.506	0.898	1.152
150	0.430	0.708	1.057	1.201
151	0.275	1.407	1.347	1.285
152	0.248	1.407	1.557	1.339
153	0.290	1.059	1.545	1.321
154	0.023	9.564	2.643	2.311
155	0.453	0.443	1.289	1.239
156	0.232	1.141	1.840	1.457
157	0.010	33.019	2.373	1.836

TABLE 2 *Continued*

N	γ_N	κ_N	σ_N	ν_N
158	0.188	1.643	1.870	1.465
159	0.289	1.014	1.577	1.339
160	0.289	1.014	1.579	1.339
161	0.366	0.746	1.379	1.268
162	0.366	0.746	1.378	1.268
163	0.282	1.097	1.563	1.329
164	0.344	0.856	1.391	1.274
165	0.274	0.953	1.719	1.389
166	0.346	0.750	1.486	1.299
167	0.221	1.396	1.771	1.408
168	0.253	1.396	1.519	1.332
169	0.249	1.462	1.511	1.325
170	0.084	4.684	1.958	1.551
171	0.117	2.661	2.091	1.608
172	1.442	−0.271	−0.640	1.032
173	0.933	0.065	0.095	0.995
174	0.458	0.614	1.041	1.185
175	0.426	0.738	1.060	1.190
176	0.420	0.738	1.104	1.190
177	0.297	1.182	1.416	1.267
178	−0.150	−3.768	2.342	1.676
179	0.453	0.595	1.110	1.183
180	0.305	0.825	1.619	1.363
181	0.484	0.388	1.200	1.223
182	0.449	0.547	1.172	1.211
183	0.385	0.647	1.376	1.266
184	0.419	0.647	1.192	1.225
185	0.279	1.098	1.550	1.346
186	0.300	1.080	1.435	1.304
187	0.177	2.068	1.717	1.422
188	0.141	2.610	1.843	1.472
189	0.143	2.610	1.834	1.459
190	0.210	1.731	1.643	1.370
191	0.286	1.149	1.484	1.312
192	0.286	1.149	1.485	1.312
193	0.348	0.862	1.355	1.270
194	0.317	0.977	1.429	1.296
195	0.347	0.914	1.306	1.258
196	0.354	0.897	1.284	1.250
197	0.406	0.721	1.181	1.222
198	0.450	0.614	1.091	1.190
199	0.494	0.514	1.003	1.169
200	0.490	0.514	1.023	1.173

TABLE 3

THE CONDITIONS IN SECTIONS 2–4 AND THE NUMBER $\#N$ OF N'S, $101 \leqslant N \leqslant 200$, FOR WHICH THESE CONDITIONS ARE SATISFIED

Condition	Theorem	Place of induction	$\#N$
$0.305 \leqslant \lambda \leqslant 1.342$	5	$x=1$	10
$\lvert\lambda - 0.426\rvert \leqslant 1.016$, $\lambda \notin \mathbb{R}$	6	$x=1$	1
$\lvert\lambda\rvert \leqslant 0.409$	6 (Remark 2)	$x=0$	10
$-0.09 \leqslant \lambda \leqslant 1.04$	7	$x=0$ and $x=1$	12
$\gamma_N \in [-13.706, 0.482]$	9(1)	$x=0$	78
$\gamma_N \notin [-0.295, 2.295]$	9(2)	$x=1$	9
$\gamma_N \leqslant 0.177$	9(3)	$x=0$ and $x=1$	20
$\lvert\kappa_N\rvert \geqslant 0.520$	11 (Corollary)	$x=0$	72
$\lvert\sigma_N\rvert \geqslant 1.990$	11 (Corollary)	$x=1$	22
$-3.051 \leqslant \kappa_N \leqslant -0.393$	12	$x=1$	4
$\nu_N \notin [0.831, 1.388]$	13	$x=1$	32

TABLE 4[a]

N	N_1	N_2	N_3	N_1/N	N_2/N	N_3/N
10,000	7,378	826	1,830	0.738	0.083	0.183
20,000	13,940	1,689	4,265	0.697	0.085	0.213
30,000	21,157	2,855	7,017	0.705	0.095	0.234
40,000	27,583	5,009	10,505	0.690	0.125	0.263
50,000	36,057	5,344	13,509	0.721	0.107	0.270
60,000	40,483	5,448	13,801	0.675	0.091	0.230
70,000	45,822	5,895	15,360	0.655	0.084	0.219
80,000	54,025	5,951	17,790	0.675	0.074	0.222
90,000	61,502	9,091	21,849	0.683	0.101	0.243
100,000	70,483	9,496	23,153	0.705	0.095	0.232
110,000	78,883	10,237	26,507	0.717	0.093	0.241
120,000	86,990	11,071	27,766	0.725	0.092	0.231
130,000	96,990	11,071	30,634	0.746	0.085	0.236
140,000	106,635	12,420	34,282	0.762	0.089	0.245
150,000	116,470	12,420	34,282	0.777	0.083	0.229
160,000	118,167	12,420	34,282	0.739	0.078	0.214
170,000	121,560	12,770	34,779	0.715	0.075	0.205
180,000	122,243	13,485	34,779	0.679	0.075	0.193
190,000	129,477	13,485	35,884	0.682	0.071	0.189
200,000	139,477	13,604	43,437	0.697	0.068	0.217
210,000	145,636	13,938	44,853	0.694	0.066	0.214
220,000	150,153	16,079	47,580	0.683	0.073	0.216
230,000	156,964	16,082	47,580	0.683	0.070	0.207
240,000	166,964	16,082	51,046	0.696	0.067	0.213
250,000	176,964	16,325	58,273	0.708	0.065	0.233

TABLE 4[a] *Continued*

N	N_1	N_2	N_3	N_1/N	N_2/N	N_3/N
260,000	186,964	16,338	60,808	0.719	0.063	0.234
270,000	193,594	20,725	66,987	0.717	0.077	0.248
280,000	200,174	21,973	73,532	0.715	0.079	0.263
290,000	210,174	21,973	75,189	0.725	0.076	0.259
300,000	218,542	21,973	75,189	0.729	0.073	0.251
310,000	222,299	21,973	75,189	0.717	0.071	0.243
320,000	226,206	21,973	75,189	0.707	0.069	0.235
330,000	231,064	22,990	77,801	0.700	0.070	0.236
340,000	231,763	22,990	77,801	0.682	0.068	0.229
350,000	236,849	22,990	77,801	0.677	0.066	0.222
360,000	239,490	22,990	77,801	0.665	0.064	0.216
370,000	248,193	22,990	77,801	0.671	0.062	0.210
380,000	258,193	22,993	86,409	0.680	0.061	0.227
390,000	268,031	25,766	96,323	0.687	0.066	0.247
400,000	275,667	28,161	99,183	0.689	0.070	0.248
410,000	277,798	31,052	100,596	0.678	0.076	0.245
420,000	282,740	31,088	100,596	0.673	0.074	0.240
430,000	283,645	31,088	100,596	0.660	0.072	0.234
440,000	290,919	34,599	107,284	0.661	0.079	0.244
450,000	300,919	36,779	115,494	0.669	0.082	0.257
460,000	310,919	36,779	117,388	0.676	0.080	0.255
470,000	320,772	36,779	117,388	0.683	0.078	0.250
480,000	330,772	36,779	118,658	0.689	0.077	0.247
490,000	340,772	36,779	123,279	0.696	0.075	0.252
500,000	348,412	38,058	127,138	0.697	0.076	0.254
510,000	350,414	39,753	128,984	0.687	0.078	0.253
520,000	357,283	42,053	135,696	0.687	0.081	0.261
530,000	357,283	42,053	135,696	0.674	0.079	0.256
540,000	357,377	42,595	135,832	0.662	0.079	0.252
550,000	360,433	45,032	138,025	0.655	0.082	0.251
560,000	370,157	51,363	147,784	0.661	0.092	0.264
570,000	380,157	51,716	157,429	0.667	0.091	0.276
580,000	390,157	51,716	164,462	0.673	0.089	0.284
590,000	400,157	51,716	166,003	0.678	0.088	0.281
600,000	410,157	51,716	166,003	0.684	0.086	0.277
610,000	420,157	51,716	166,003	0.689	0.085	0.272
620,000	430,157	51,716	166,003	0.694	0.083	0.268
630,000	438,965	51,716	166,003	0.697	0.082	0.264
640,000	448,965	51,787	170,225	0.702	0.081	0.266
650,000	453,663	56,339	175,101	0.698	0.087	0.269
660,000	453,663	56,339	175,101	0.687	0.085	0.265
670,000	461,650	56,339	175,101	0.689	0.084	0.261
680,000	471,573	56,339	175,101	0.694	0.083	0.258
690,000	481,573	56,339	175,101	0.698	0.082	0.254
700,000	491,573	56,339	175,101	0.702	0.081	0.250

TABLE 4[a] *Continued*

N	N_1	N_2	N_3	N_1/N	N_2/N	N_3/N
710,000	501,573	56,339	175,101	0.706	0.079	0.247
720,000	511,573	56,339	175,716	0.711	0.078	0.244
730,000	521,573	56,339	180,266	0.715	0.077	0.247
740,000	531,573	56,339	182,462	0.718	0.076	0.247
750,000	536,580	56,339	184,016	0.715	0.075	0.245
760,000	536,761	56,339	184,016	0.706	0.074	0.242
770,000	543,454	56,339	184,016	0.706	0.073	0.239
780,000	550,099	56,823	184,568	0.705	0.073	0.237
790,000	555,841	58,074	186,454	0.704	0.074	0.236
800,000	561,827	58,522	187,465	0.702	0.073	0.234
810,000	565,314	59,302	188,638	0.698	0.073	0.233
820,000	566,405	59,302	188,638	0.691	0.072	0.230
830,000	576,140	59,302	188,638	0.694	0.071	0.227
840,000	586,140	59,302	188,638	0.698	0.071	0.225
850,000	596,140	59,302	188,638	0.701	0.070	0.222
860,000	606,140	59,302	188,638	0.705	0.069	0.219
870,000	615,461	59,302	188,638	0.707	0.068	0.217
880,000	619,294	59,302	188,638	0.704	0.067	0.214
890,000	626,466	59,302	188,638	0.704	0.067	0.212
900,000	636,466	59,302	188,638	0.707	0.066	0.210
910,000	643,208	59,302	188,638	0.707	0.065	0.207
920,000	650,172	59,302	188,638	0.707	0.065	0.205
930,000	660,172	59,302	188,638	0.710	0.064	0.203
940,000	670,172	59,302	188,638	0.713	0.063	0.201
950,000	680,172	59,302	188,638	0.716	0.062	0.199
960,000	690,172	59,302	190,614	0.719	0.062	0.199
970,000	700,171	62,476	200,552	0.722	0.064	0.207
980,000	704,529	64,502	203,342	0.719	0.066	0.208
990,000	704,529	64,502	203,342	0.712	0.065	0.205
1,000,000	704,529	64,502	203,342	0.705	0.065	0.203

[a]Compare Theorem 9.

$$N_1 := \#\{M \leqslant N, \gamma_M \in [-13.706, 0.482]\},$$

$$N_2 := \#\{M \leqslant N, \gamma_M \notin [-0.295, 2.295]\},$$

$$N_3 := \#\{M \leqslant N, \gamma_M \leqslant 0.177\}.$$

A closer examination of the values γ_N shows that for $10^4 \leqslant N \leqslant 10^6$:

each interval $[N, N+N^{1-0.21}]$ contains at least one M such that $\gamma_M \in [-13.706, 0.482]$;

each interval $[N, N+N^{1-0.11}]$ contains at least one M such that $\gamma_M \notin [-0.295, 2.295]$;

each interval $[N, N+N^{1-0.12}]$ contains at least one M such that $\gamma_M \leqslant 0.177$.

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